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Analog Communication

Electrical Engineering

Comprehensive Theory *with* Solved Examples

Civil Services Examination

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CHAPTER 2

Basics of Signal and System

Introduction

Just as a carpenter requires proper set of tools before he can sit down to make a piece of furniture, in a similar manner a communication engineer needs to know about signals before he can start the process of learning communication.

2.1 Signal and System

The communication technology can be conveniently broken down into three interacting parts.

- Signal processing operations performed.
- The device that performs these operations.
- The underlining physics.

Thus to study the basic form of modulation and signal processing used in the communication it will be fruitful to have a quick review of the concepts of signal and system.

2.1.1 Some Basic Signals

It will be very helpful to study some signals before hand, so that the analysis of the communication system becomes easier. Some important and frequently used signals and their properties are mentioned in this section.

The Impulse Signal

Impulse function is not a function in its strict sense. It is a distributed or generalized function. A generalized function is defined in terms of its effect on other function. The unit impulse function is generalised as any function that follow the following condition:

1. Impulse signal (Dirac delta function):

$$\delta(t) = \begin{cases} \infty ; & t = 0 \\ 0 ; & t \neq 0 \end{cases}$$

and

$$\int_{-\infty}^{\infty} \delta(t)dt = 1$$

2. Unit impulse signal:

$$\delta[n] = \begin{cases} 1 ; & n = 0 \\ 0 ; & n \neq 0 \end{cases}$$

Properties of Impulse Function

1. Product property

$$x(t)\,\delta(t) = x(0)\,\delta(t)$$

Similarly, $$x(t)\,\delta(t-\alpha) = x(\alpha)\,\delta(t-\alpha)$$

2. Shifting property

$$\int_{-\infty}^{\infty} x(t)\,\delta(t)dt = x(0)$$

Similarly,

$$\int_{-\infty}^{\infty} x(t)\,\delta(t-\alpha)dt = x(\alpha)$$

$$\int_{-\infty}^{\infty} \delta(t)dt = 1$$

3. Scaling property

$$\delta(\alpha t) = \frac{1}{|\alpha|}\delta(t)$$

Example 2.1 Find the impulse function form if $x(t) = 4t^2\,\delta(2t-4)$, where $x(t)$ is an arbitrary signal.

***Solution*:**

$$x(t) = 4t^2\,\delta(2t-4)$$
$$= 4t^2\,\delta\{2(t-2)\}$$
$$= 4t^2 \cdot \frac{1}{2}\delta(t-2) \qquad \text{... from scaling property}$$
$$= 2t^2\,\delta(t-2)$$

Now, from product property we have,

$$x(t)\,\delta(t-\alpha) = x(\alpha)\,\delta(t-\alpha)$$

So, $$x(t) = 2t^2\Big|_{t=2} \cdot \delta(t-2) = 8\,\delta(t-2)$$

Do you know? Impulse signals do not occur naturally but they are important functions providing a mathematical frame work for the representation of various processes and signals. These come under a special class of functions known as generalized functions.

Gate Function/Rectangular Pulse

Let us consider a rectangular pulse as shown in figure below:

$$x(t) = A\,\text{rect}(t) = \begin{cases} A, & \text{for } \frac{-1}{2} < t < \frac{1}{2} \\ 0 & \text{otherwise} \end{cases}$$

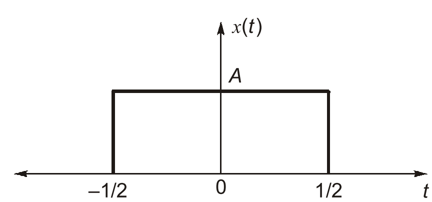


Figure-2.1

$$x(t) = A\,rect\left(\frac{t}{\tau}\right) = \begin{cases} A, & \text{for } \frac{-\tau}{2} < t < \frac{\tau}{2} \\ 0 & \text{otherwise} \end{cases}$$

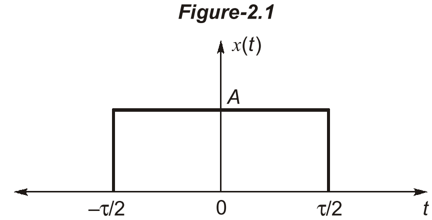


Figure-2.2

Step Signal

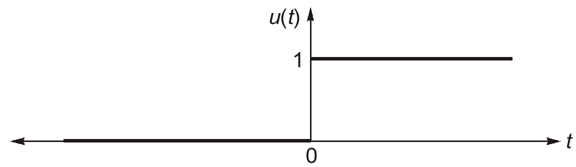


Figure-2.3 : *Continuous-time version of the unit-step function of unit amplitude*

The continuous-time version of the unit-step function is defined by

$$u(t) = \begin{cases} 1; & t > 0 \\ 0; & t < 0 \end{cases}$$

NOTE

- Figure depicts the unit-step function $u(t)$. It is said to exhibit discontinuity at $t = 0$, since the value of $u(t)$ changes instantaneously from 0 to 1 when $t = 0$. It is for this reason that we have left out the equal sign in equation; that is $u(0)$ is undefined.
- Unit step function denote sudden change in real time and a frequency or phase selectivity in frequency domain.

There is one more definition of unit step function.

$$u(t) = \begin{cases} 0 & ; t < 0 \\ 1/2 & ; t = 0 \\ 1 & ; t > 0 \end{cases}$$

Properties of Unit-Step Function

1. $u(t - t_0) = [u(t - t_0)]^2 = u[u(t - t_0)]^k$, with k being any positive integer.
2. $u(at - t_0) = u\left(t - \dfrac{t_0}{a}\right); a > 0$
3. $\delta(t) = \dfrac{d}{dt} u(t)$
4. $u(t) = \displaystyle\int_{-\infty}^{t} \delta(\tau)\, d\tau = \begin{cases} 1, & t > 0 \\ 0, & t < 0 \end{cases}$

Do you know? The unit-step function $u(t)$ may also be used to construct other discontinuous waveforms. The value at $t = 0$ gives rise to Gibb's phenomenon when unit step function is constructed by sinusoidal signals.

Sampling/Interpolating/Sinc Function

The function $\dfrac{\text{"}\sin \pi x\text{"}}{\pi x}$ is the **"sine over argument"** function and it is denoted by "sinc (x)". It is also known as **"filtering function"**.

Mathematically,

$$\text{sinc}(x) = \frac{\sin \pi x}{\pi x} = Sa(\pi x)$$

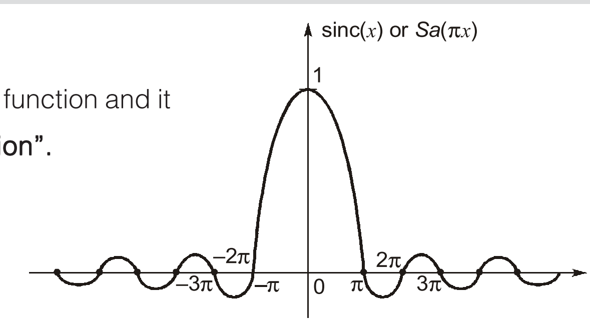


Figure-2.4 : *Sinc Function*

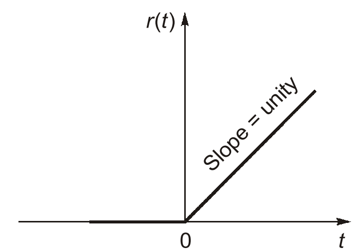
Do you know? Just like impulse function sinc (x) is also a conceptual function since it can not be realized.

The Unit-Ramp Function

The ramp function $r(t)$ is a linearly growing function for positive values of independent variable t. The ramp function shown in figure is defined by

$$r(t) = \begin{cases} t, & t \geq 0 \\ 0, & t < 0 \end{cases}$$

or $$r(t) = tu(t)$$



The ramp function is obtained by integrating the unit step function

$$\int_{-\infty}^{t} u(\tau)d\tau = r(t)$$

The relationship between the impulse, step and ramp signals are represented below:

Remember: Relationship between impulse, step and ramp signals

$$\delta(t) \xrightarrow{\text{Integrate}} u(t) \xrightarrow{\text{Integrate}} r(t)$$

$$r(t) \xrightarrow{\text{Differentiate}} u(t) \xrightarrow{\text{Differentiate}} \delta(t)$$

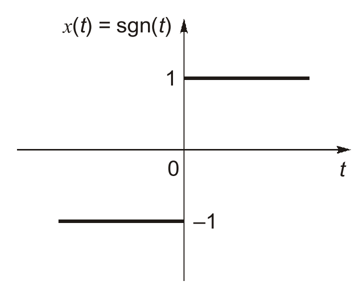
Unit Signum Function

The unit signum function shown in figure is defined as follows

$$\text{sgn}(t) = \begin{cases} 1 & t > 0 \\ 0 & t = 0 \\ -1 & t < 0 \end{cases}$$

This function can be expressed in terms of unit step function as

$$\text{sgn}(t) = -1 + 2u(t)$$



Unit Signum Function

2.1.2 Signal-Classification

Continuous-Time and Discrete-Time signals

The signals that are defined at each instant of time are known as continuous time signals. However, if the signals are defined only at certain time instants, it is called as discrete-time signals.

Based upon above discussion, four combinations are possible:

- Continuous time continuous amplitude signal (Analog signal)
- Continuous time discrete amplitude signal (Quantized signal)
- Discrete time continuous amplitude signal (Sampled signal)
- Discrete time discrete amplitude signal (Digital signal)

Analog and Digital Signal

If the amplitude of the signal can take all possible values in its dynamic range, it is called as analog signal. On the other hand, a digital signal is one whose amplitude take some specific values in its dynamic range.

Periodic and Aperiodic Signals

A signal is said to be periodic if it repeats itself after a certain time interval. For a signal to be periodic, it must satisfy the following condition.

1. It should exist for all values of 't'.
2. $x(t) = x(t + T)$, where T is the least value after which the signal repeats itself.
3. The value of T should be a fixed positive constant.

'T' is referred as fundamental period.

Any signal which do not follow these conditions are termed as aperiodic signal.

NOTE

Periodicity of Signal $x_1(t) + x_2(t)$:

A signal $x(t)$ that is a linear combination of two periodic signals, $x_1(t)$ with fundamental period T_1 and $x_2(t)$ with fundamental period T_2 as follows:

$$x(t) = x_1(t) + x_2(t)$$

is periodic if, $\dfrac{T_1}{T_2} = \dfrac{m}{n}$ = a rational number

Period of $x(t)$, $T = nT_1 = mT_2$

or, $T = \text{LCM}(T_1, T_2)$

Deterministic and Random Signal

A signal is said to be deterministic, if they can be completely represented by a mathematical expression at any instance of time. Signals, which cannot be represented by any mathematical expression is called random signal.

Note: For analysis purpose random signal can also be approximated by their statistical property.

Energy Signals and Power Signals

$x(t)$ is an energy signal if

$0 < E < \infty$ and $P = 0$

where 'E' is the energy and 'P' is the power of the signal $x(t)$.

For a continuous-time signal (CTS),

$$E = \int_{-\infty}^{\infty} |x(t)|^2 \, dt$$

For an energy signal, energy is finite while power is zero.

NOTE

If $x(t) \longrightarrow E$, [where, E is energy of $x(t)$]

then $x\left(\dfrac{t}{\alpha}\right) \longrightarrow \alpha E$

$x(\alpha t) \longrightarrow \dfrac{E}{\alpha}$

$ax(t) \longrightarrow a^2 E$

$x(t)$ is a Power Signal if

if, $0 < P < \infty$ and $E = \infty$

where E = Energy of signal $x(t)$

P = Power of signal $x(t)$

Almost all the practical periodic signals are **"power signals"**, since their average power is finite and non-zero.

For a CTS, the average power of a signal $x(t)$ is,

$$P_x = \lim_{T \to \infty} \frac{1}{T} \int_{-T/2}^{T/2} |x(t)|^2 \, dt = \lim_{T \to \infty} \frac{1}{2T} \int_{-T}^{T} |x(t)|^2 \, dt$$

NOTE

- If $x(t) = A\cos\omega t$ or $A\sin\omega t$, then $P_x = A^2/2$
- If $x(t) = Ae^{\pm j\omega t} \Rightarrow P_x = A^2$
- If $x(t) = A \Rightarrow P_x = A^2$
- If $x(t) \longrightarrow P$, then $x\left(\frac{t}{\alpha}\right) \longrightarrow P$

  $x(\alpha t) \longrightarrow P$ and $ax(\alpha t) \longrightarrow a^2P$
- For an **unit step signal**, $x(t) = u(t)$ and $P_x = \frac{1}{2}$

Energy Signal	Power Signal
1. The total energy is obtained using $E = \lim_{T\to\infty}\int_{-T}^{T}\lvert x(t)\rvert^2 dt$	The average power is obtained $P = \lim_{T\to\infty}\int_{-T}^{T}\frac{1}{2T}\lvert x(t)\rvert^2 dt$
2. For the energy signal, $0 < E < \infty$, and the average power $P = 0$	For the power signal, $0 < P < \infty$, and the energy $E = \infty$.
3. Non-periodic and finite duration signals are in general energy signals.	Periodic signals are power signals. However, all power signals need not be periodic.
4. Energy signals are time limited.	Power signals exist over infinite time.

Table-2.1

2.2 Time Domain and Frequency Domain Representation of a Signal

A signal $x(t)$ can be represented in terms of relative amplitude of various frequency components present in signal. This is possible by using exponential Fourier series. This is a frequency domain representation of the signal. The time domain representation specifies a signal value at each instant of time. This means that a signal $x(t)$ can be specified in two equivalent ways:

(i) Time domain representation; where $x(t)$ is represented as a function of time. Graphical time domain representation is termed as *waveform.*

(ii) The frequency domain representation; where the signal is represented graphically in terms of its frequency graphical frequency domain representation is termed as spectrum.

Any of the above two representations uniquely specifies the function, i.e. if the signal is specified in time domain, we can determine its spectrum. Conversely, if the spectrum is specified, we can determine the corresponding time domain signal. In order to determine the function in frequency domain, it is necessary that both amplitude spectrum and phase spectrum are specified.

Remember

- In many cases, the spectrum is either real or imaginary, as such, only an amplitude plot is enough as all frequency components have identical phase relation.
- We use both the conventions depending upon the problem we are studying.
- If we want to analyze the signal at our perspective, it is convenient to see signal in its time domain form, but if we want to process the signal through an LTI system, the frequency domain approach becomes much fruitful.

2.2.1 Decomposition of Signals

It will be fruitful for us if we can devise some technique that will allow us to break any unknown signal into some standard and known signal set. There are two methods of doing this

(1) Any signal can be broken down into an infinite set of impulse signals. This process leads to the time domain approach of signal and system.

(2) It can be broken down into an infinite set of orthogonal signals. This leads to frequency domain description of signal and system.

Misconception: Not any representation of signal as set of impulse or exponential function is considered as signal decomposition.

Consider a periodic signal $x(t) = \begin{cases} 1\,; & 0 \le t \le 1 \\ -2\,; & 1 < t < 2 \end{cases}$ with time period $T = 2$.

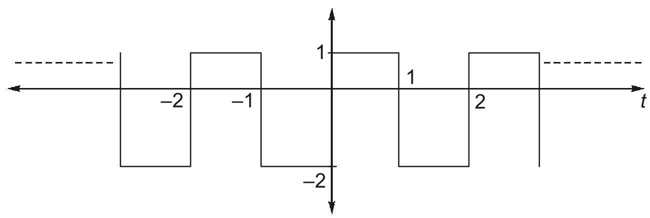


Figure-2.5

The derivative of this signal is related to the impulse train $g(t) = \sum_{k=-\infty}^{\infty} \left[a_k \delta(t-k) + b_k \delta(t-2k)\right]$ with period $T = 2$. [where, $a_k = 3$, $b_k = -3$]

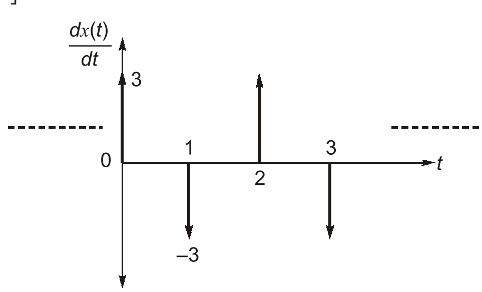


Figure-2.6

NOTE: The problem here is we have to define a differentiator in order to represent the time domain signal as set of impulses. Thus we need an alternative way of representing the signal.

2.3 Signals Versus Vectors

There is a strong connection between signals and vectors. Signals that are defined for only a finite number of time instants (say N) can be written as vectors (of dimension N). Thus, consider a signal $g(t)$ defined over a closed time interval $[a, b]$. Let us pick N points uniformly on the time interval $[a, b]$ such that

$$t_1 = a,\ t_2 = a + \epsilon,\ t_3 = a + 2\epsilon,\ t_N = a + (N-1)\epsilon,\ \epsilon = \frac{b-a}{N-1}$$

Then we can write a signal vector g as an N-dimensional vector

$$g = [g(t_1)\ g(t_2)\ \ldots\ g(t_N)]$$

This relationship clearly shows that continuous time signals are straight forward generalizations of finite dimension vectors. Thus, basic definitions and operations in a vector space can be applied to continuous time signals as well. In a vector space, we can define the inner (dot or scalar) product of two real-valued vectors x and g as

$$< x, g > \; = \|g\| \cdot \|x\| \cos\theta$$

When θ is the angle between vectors x and g.

By using this definition, we can express $\|x\|$, the length (norm) of a vector x as

$$\|x\|^2 = < x, x >$$

Remember: This concept forms the basis of digital communication system.

2.3.1 Decomposition of a Signal and Signal Components

The concepts of vector component and orthogonality can be directly extended to continuous time signals. Consider the problem of approximating a real signal $g(t)$ in terms of another real signal $x(t)$ over an interval $[t_1, t_2]$.

$$g(t) \approx ex(t)$$

$$e = \frac{\int_{t_1}^{t_2} g(t)x(t)dt}{\int_{t_1}^{t_2} x^2(t)\,dt} = \frac{1}{E_x}\int_{t_1}^{t_2} g(t)\,x(t)dt \quad \text{(where, } E_x = \text{Energy of signals)}$$

NOTE

If a signal $g(t)$ is approximated by another signal $x(t)$ as

$$g(t) = ex(t)$$

then the optimum value of e that minimizes the energy of the error signal in this approximation is given by above equation.

2.4 Orthogonal Signal Set

In this section we show a way of representing a signal as a sum of orthogonal set of signals. Infact, the signals in this orthogonal set form a basis for the specific signal space. Here again we can benefit from the insight gained from a similar problem in vectors. We know that a vector can be represented as a sum of orthogonal vectors, which form the coordinate system of a vector space.

2.4.1 Orthogonal Signal Space

We continue with signal approximation problem, using clues and insights developed for vector approximation. We define orthogonality of a signal set $x_1(t), x_2(t), \ldots x_n(t)$ over a time domain Θ.

$$\int x_m(t)x_n^*(t)dt = \begin{cases} 0 & m \neq n \\ E_n & m = n \end{cases}$$

If all signal energies are equal to unity $E_n = 1$, then the set is normalized and is called an orthonormal set. An orthogonal set can always be normalized by dividing $x_n(t)$ by $\sqrt{E_n}$ for all n. A signal $g(t)$ over the time domain Θ can be represented by a set of N mutually orthogonal signals $x_1(t), x_2(t), \ldots\, x_N(t)$:

$$g(t) \simeq C_1 x_1(t) + C_2 x_2(t) + \ldots + C_N x_N(t)$$

$$= \sum_{n=1}^{N} C_n x_n(t)$$

It can be shown that E_n, the energy of the error signal $e(t)$ in this approximation, is minimized if we choose

$$C_n = \frac{\int_{t\in\Theta} g(t)x_n^*(t)dt}{\int_{t\in\Theta} |x_n(t)|^2\,dt} = \frac{1}{E_n}\int g(t)x_n^*(t)dt \quad n = 1, 2, \;\ldots..\; N$$

NOTE : If the orthogonal set is complete, then the error energy $E_n \to 0$.

Remember: Orthogonal signal set forms the basis signals set for the representation of signals just like the coordinate axis are used in maps to represent a point.

2.5 The Fourier Series

Let $g_p(t)$ represent a periodic signal with period to. With the help of Fourier series, we are able to resolve the signal into infinite sum of sine and cosine signals. An alternative way of saying this definition can be we break up the periodic signal $g_p(t)$ into an infinite sum of harmonics which helps us to define the signal in the frequency domain.

There are two ways to represent the Fourier series.

(1) Trigonometric Fourier series

(2) Exponential Fourier series

2.5.1 Trigonometric Form of Fourier Series

The expression may be expressed as.

$$g_p(t) = a_0 + \sum_{n=1}^{\infty}\left[a_n \cos\left(\frac{2\pi nt}{T_0}\right) + b_n \sin\left(\frac{2\pi nt}{T_0}\right)\right]$$

The above equation is termed as **Synthesis Equation**. Thus the periodic signal $g_p(t)$ is now represented as set of orthogonal signal.

Where the coefficients a_n and b_n represents the unknown amplitudes of the cosine and sine forms, respectively. The quantity with n/T_0 represents the n^{th} harmonic of the fundamental frequency $f_0 = \frac{1}{T_0}$.

Now, we need to calculate the coefficient a_n, b_n and a_0. The equation which are used to calculate these value are known as **Analysis Equation**. The equations for the trigonometric Fourier series are given below.

$$a_0 = \frac{1}{T_0}\int_0^{T_0} g_p(t)dt$$

$$a_n = \frac{2}{T_0}\int_0^{T_0} g_p(t)\cos\left(\frac{2\pi nt}{T_0}\right)dt$$

$$b_n = \frac{2}{T_0}\int_0^{T_0} g_p(t)\sin\left(\frac{2\pi nt}{T_0}\right)dt$$

2.5.2 Exponential Form of Fourier Series

In the above expression of trigonometric Fourier series the orthogonal set used for the creation of the periodic signal were sine and cosine terms. In the similar way we can represent the periodic signal in terms of an infinite set of orthogonal complex exponential signals.

Synthesis equation

$$g_p(t) = \sum_{n=-\infty}^{\infty} C_n e^{jn\omega_o t}\ ; \quad n = 0,\ 1,\ 2,\ 3......$$

Analysis equation

$$C_n = \frac{1}{T_0}\int_0^T g_p(t)e^{-jn\omega_o t}\,dt = \frac{1}{T_0}\int_{-T_0/2}^{T_0/2} g_p(t)e^{\frac{-j2\pi nt}{T_0}}\,dt$$

NOTE :
- C_n and C_{-n} are in complex-conjugate pair when the signal $g_p(t)$ is a real signal i.e. $C_n = C_{-n}^*$
- C_ns are the spectral amplitudes of the spectral component $C_n\, e^{j2\pi n f_0 t}$.

2.5.3 Frequency Spectrum of Non-sinusoidal Wave

Amplitude of wave is 'A' and repetition rate is $\omega/2\pi$ per second, then.

(a) Square wave: $g_p(t) = \dfrac{4A}{\pi}\left(\cos\omega t - \dfrac{1}{3}\cos 3\omega t + \dfrac{1}{5}\cos 5\omega t - \dfrac{1}{7}\cos 7\omega t + ...\right)$

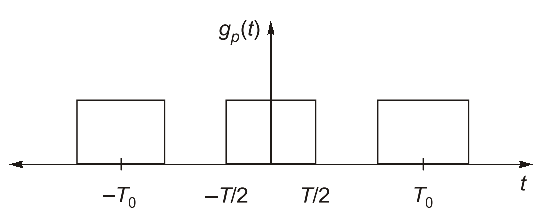
(b) Triangular wave: $g_p(t) = \dfrac{4A}{\pi^2}\left(\cos\omega t - \dfrac{1}{9}\cos 3\omega t + \dfrac{1}{25}\cos 5\omega t + ...\right)$

(c) Sawtooth wave: $g_p(t) = \dfrac{2A}{\pi}\left(\sin\omega t - \dfrac{1}{2}\sin 2\omega t + \dfrac{1}{3}\sin 3\omega t - \dfrac{1}{4}\sin 4\omega t + ...\right)$

NOTE

From the analysis equation in trigonometric Fourier series we conclude that:
- The trigonometric Fourier series of an even function of time contains only D.C. term and cosine terms.
- The trigonometric Fourier series of an odd function of time contains only sine terms.

Example 2.2 Given a periodic signal $g_p(t)$ as shown the figure below.



Find the complex Fourier coefficient C_n.

***Solution*:**

$$g_p(t) = \begin{cases} A & -T/2 \le t \le T/2 \\ 0 & \text{for the remainder of the period} \end{cases}$$

Now, $$C_n = \frac{1}{T_0}\int_{-T/2}^{T/2} A\exp\left(\frac{-i2\pi nt}{T_0}\right)dt$$

$\therefore$ $$C_n = \frac{A}{n\pi}\sin\left(\frac{n\pi T}{T_0}\right)\ ; \quad n = 0,\ \pm 1,\ \pm 2\ ...$$

Example 2.3 The complex exponential Fourier series representation of a signal $f(t)$ over the interval $(0, T)$ is

$$f(t) = \sum_{n=-\infty}^{+\infty} \frac{3}{4+(n\pi)^2} e^{jn\pi t}$$

Determine:

(a) the numerical value of T;

(b) the numerical value of A, if one of the components of $f(t)$ is $A \cos 3\pi t$.

Solution:

The standard representation of complex Fourier series of signal $f(t)$ in given as

$$f(t) = \sum_{n=-\infty}^{\infty} F_n e^{j2\pi nt/T} = \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t} \quad \text{...(i)}$$

where,
$$a_k = F_n = \frac{1}{T}\int_0^T f(t)e^{-jk\omega_0 t}dt \quad \text{...(ii)}$$

Given that,
$$f(t) = \sum_{n=-\infty}^{\infty} \frac{3}{4+(n\pi)^2} e^{jn\pi t} \quad \text{...(iii)}$$

(a) Comparing, equation (iii) with equation (i) we get,

$$\Rightarrow \quad \frac{2}{T} = 1$$

$$\Rightarrow \quad T = 2$$

(b) At $n = 3$, one component of $f(t)$ is

$$f_1(t) = \frac{3}{4+(3\pi)^2} e^{j3\pi t} = \frac{3}{4+(3\pi)^2}[\cos 3\pi t + j\sin 3\pi t]$$

Similarly at $n = -3$, another component of $f(t)$ is

$$f_2(t) = \frac{3}{4+(3\pi)^2} e^{-j3\pi t}$$

$$= \frac{3}{4+(3\pi)^2}[\cos 3\pi t - j\sin 3\pi t]$$

$$\therefore \quad f_1(t) + f_2(t) = 2\cdot\frac{3}{4+(3\pi)^2}\cos 3\pi t$$

$$= \frac{6}{4+(3\pi)^2}\cos 3\pi t$$

Comparing with A cos $3\pi t$, we have,

$$A = \frac{6}{4+9\pi^2} = 6.464 \times 10^{-2}$$

"Parseval's theorem" for power signal $x(t)$

It states that power of a signal $x(t)$ may be defined in terms of its Fourier series coefficients. For the trigonometric Fourier series,

Since,
$$x(t) = C_0 + \sum_{n=1}^{\infty} C_n \cos(n\omega_o t + \theta_n)$$

$$C_n = \sqrt{a_n^2 + b_n^2} \; ; \; \theta_n = \tan^{-1}\left(-\frac{b_n}{a_n}\right)$$

then power of $x(t)$ is given by,

$$P_x = C_o^2 + \frac{1}{2}\sum_{n=1}^{\infty} C_n^2$$

For the **exponential Fourier series**,

since,
$$x(t) = C_o + \sum_{\substack{n=-\infty \\ (n \neq 0)}}^{\infty} C_n e^{jn\omega_o t}$$

then power of $x(t)$ is given by,

$$P_x = \sum_{n=-\infty}^{\infty} |C_n|^2$$

For real $x(t)$,
$$|C_n| = |C_{-n}|$$

$\therefore$
$$P_x = C_0^2 + 2\sum_{n=1}^{\infty} |C_n|^2$$

"Power Spectral Density" (PSD) may be treated as average power per unit bandwidth. It is generally denoted by $S(\omega)$.

$$S(\omega) = \lim_{\tau \to \infty} \frac{|X_\tau(\omega)|^2}{\tau} = \frac{dP(\omega)}{d\omega}$$

$\therefore$ Total power of signal $x(t) = P_T$

$\Rightarrow$
$$P_T = \frac{1}{2\pi}\int_{-\infty}^{\infty} S(\omega)\,d\omega = \int_{-\infty}^{\infty} S(f)\,df$$

Also,
$$\text{PSD} = S(\omega) = 2\pi \sum_{n=-\infty}^{\infty} |C_n|^2 \cdot \delta(\omega - n\omega_o)$$

where,
$$C_n = \frac{1}{T}\int_o^T x(t)e^{-jn\omega_o t}\,dt$$

Since, $|X_\tau(\omega)|^2 = |X_\tau(-\omega)|^2$, then average power is expressed as,

$$P_{av} = \frac{1}{\pi}\int_o^{\infty} S(\omega)\,d\omega$$

Remember

- Infinite duration signals are power signals and also power is always additive in nature.
- $u(t)$ is a power signal but it is not a periodic signal.
- All the periodic signals are always power-signals but inverse is always not true.

2.6 Fourier Transforms of Signals

A periodic signal with period infinity is called aperiodic signal which has total finite energy, its frequency domain representation will be a continuous spectrum obtained from the Fourier transform. So, we can say that an aperiodic signal $x(t)$ can be represented by a Fourier integral (rather than Fourier series).

Let, $x(t) \xrightleftharpoons{FT} X(\omega)$ then

$$X(\omega) = T \cdot a_k = \int_{-\infty}^{\infty} x(t) e^{-j\omega t} dt$$

where, a_k = Fourier series coefficient

$$= \frac{1}{T_0} \int_{-T_0/2}^{T_0/2} x(t)\, e^{-jk\omega_0 t}\, dt$$

as $T_0 \to \infty$

also, $a_k = \frac{1}{T} X(\omega) \Big|_{\text{at } \omega = k\omega_0}$

where, $\omega_0 = \frac{2\pi}{T}$

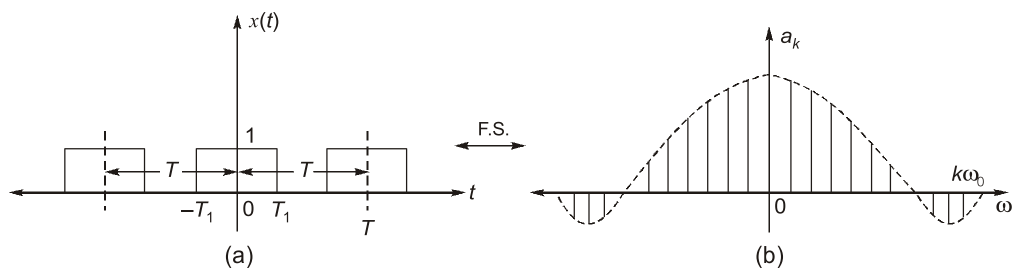


Figure-2.7

But when, $T \to \infty$

then,

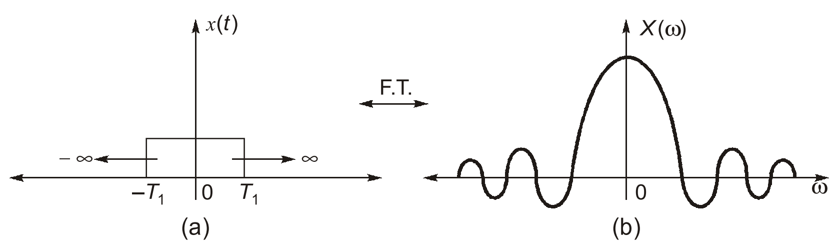


Figure-2.8 : *(a) Aperiodic signal (b) Continuous nature*

The original signal $x(t)$ is recovered by using the formula for inverse Fourier transform as:

$$x(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} X(\omega)\, e^{j\omega t}\, d\omega$$

NOTE : $x(t)$ and $X(\omega)$ form a Fourier transform pair.

In general, the Fourier transform $X(\omega)$ is a complex function of frequency "ω" so,

$$X(\omega) = |X(\omega)| e^{j\theta(\omega)}$$

where, $|X(\omega)|$ = Continuous amplitude spectrum of $x(t)$.

$\theta(\omega)$ = Continuous phase spectrum of $x(t)$.

2.6.1 Fourier transform of rectangular pulse/gate function

Let, $$x(t) = A\,\text{rect}\left(\frac{t}{\tau}\right) = \begin{cases} A, \text{ for } \dfrac{-\tau}{2} < t < \dfrac{\tau}{2} \\ 0, \text{ elsewhere} \end{cases}$$

$\therefore$ $$\text{F.T. of } x(t) = X(\omega) = \int_{-\infty}^{\infty} x(t) e^{-j\omega t}\, dt$$

$\therefore$ $$X(\omega) = A\int_{-\tau/2}^{\tau/2} e^{-j\omega t}\, dt = \frac{-A}{j\omega}\left[e^{-j\omega t}\right]_{-\tau/2}^{\tau/2}$$

$\therefore$ $$X(\omega) = \frac{2A}{\omega}\left[\frac{e^{j\omega\tau/2} - e^{-j\omega\tau/2}}{2j}\right] = \frac{2A}{\omega}\sin\left(\frac{\omega\tau}{2}\right)$$

$\therefore$ $$X(\omega) = A\tau\left\{\frac{\sin(\omega\tau/2)}{(\omega\tau/2)}\right\}$$

$$X(\omega) = A\tau\, Sa\left(\frac{\omega\tau}{2}\right) = A\tau\left[\frac{\sin(\pi f\tau)}{\pi f\tau}\right]$$

$$= A\tau\,\text{sinc}\,(f\tau)$$

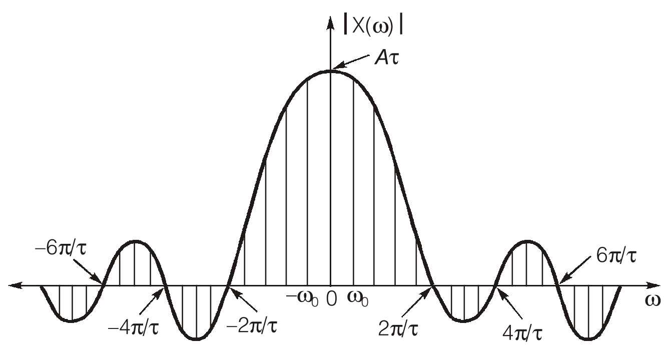
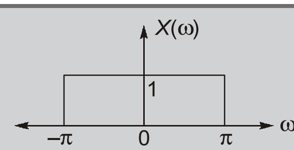


Figure-2.9

NOTE

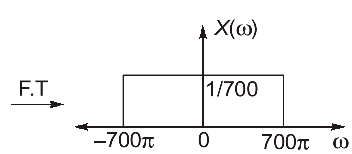
$$\text{sinc } t = \frac{\sin \pi t}{\pi t} \xrightleftharpoons{F.T.}$$



Example 2.4 Find the Fourier transform of sinc 700*t*.

Solution:

Let $$x(t) = \text{sinc}(700\, t) = \frac{\sin 700\pi t}{700\,\pi t} = \left(\frac{1}{700}\right)\cdot\frac{\sin 700\,\pi t}{\pi t}$$



2.6.2 Important Fourier Transform

S. No.	$x(t)$	$X(\omega)$	$X(f)$	Comment
1.	$e^{-at}u(t)$	$\frac{1}{a+j\omega}$ $a>0$	$\frac{1}{a+j(2\pi f)}$	Asymmetric, complex
2.	$e^{at}u(-t)$	$\frac{1}{a-j\omega}$ $a>0$	$\frac{1}{a-j(2\pi f)}$	Asymmetric, complex
3.	$e^{-a\|t\|}$	$\frac{2a}{a^2+\omega^2}$, $a>0$	$\frac{2a}{a^2+(2\pi f)^2}$	Real and even symmetric
4.	$te^{-at}u(t)$	$\frac{1}{(a+j\omega)^2}$, $a>0$	$\frac{1}{(a+j2\pi f)^2}$	Multiplication of t
5.	$t^n e^{-at}u(t)$	$\frac{n!}{(a+j\omega)^{n+1}}$, $a>0$	$\frac{n!}{(a+j2\pi f)^{n+1}}$	Multiplication of t^n
6.	$\delta(t)$	1	1	Real and even symmetric
7.	A	$2\pi A\delta(\omega)$	$A\delta(f)$	Real and even symmetric
8.	$e^{j\omega_0 t}$	$2\pi\delta(\omega-\omega_0)$	$\delta(f-f_0)$	Frequency shifting
9.	$\cos\omega_0 t$	$\pi[\delta(\omega-\omega_0)+\delta(\omega+\omega_0)]$	$\frac{1}{2}[\delta(f-f_0)+\delta(f+f_0)]$	Used in modulation property
10.	$\sin\omega_0 t$	$j\pi[\delta(\omega+\omega_0)-\delta(\omega-\omega_0)]$	$\frac{j}{2}[\delta(f+f_0)+\delta(f-f_0)]$	Used in modulation property
11.	$u(t)$	$\frac{1}{j\omega}+\pi\delta(\omega)$	$\frac{1}{j2\pi f}+\frac{\delta(f)}{2}$	Unit step function
12.	$\text{sgn}(t)$	$\frac{2}{j\omega}$	$\frac{1}{j\pi f}$	Imaginary and odd symmetric
13.	$\cos\omega_0 t\, u(t)$	$\frac{\pi}{2}[\delta(\omega-\omega_0)+\delta(\omega+\omega_0)]+\frac{j\omega}{\omega_0^2-\omega^2}$	$\frac{1}{2}[\delta(f-f_0)+\delta(f+f_0)]+\frac{j2\pi f}{(2\pi f_0)^2-(2\pi f)^2}$	
14.	$\sin\omega_0 t\, u(t)$	$j\frac{\pi}{2}[\delta(\omega+\omega_0)-\delta(\omega-\omega_0)]+\frac{\omega_0}{\omega_0^2-\omega^2}$	$\frac{j}{2}[\delta(f+f_0)-\delta(f-f_0)]+\frac{2\pi f_0}{(2\pi f_0)^2-(2\pi f)^2}$	
15.	$e^{-at}\sin\omega_0 t\, u(t)$	$\frac{\omega_0}{(a+j\omega)^2+\omega_0^2}$, $a>0$	$\frac{2\pi f_0}{(a+j2\pi f)^2+2\pi f_0^2}$, $a>0$	Decaying sin function
16.	$e^{-at}\cos\omega_0 t\, u(t)$	$\frac{a+j\omega}{(a+j\omega)^2+\omega_0^2}$, $a>0$	$\frac{a+j2\pi f}{(a+j2\pi f)^2+2\pi f_0^2}$, $a>0$	Decaying cosine function
17.	$\text{rect}(t/\tau)$	$\tau\text{Sa}\left(\frac{\omega\tau}{2}\right)$	$\tau\text{sinc}(f\tau)$	Rectangular function
18.	$W\text{sinc}(Wt)$	$\text{rect}\left(\frac{\omega}{2\pi W}\right)$	$\text{rect}\left(\frac{f}{W}\right)$	Sinc function
19.	$\sum_{n=-\infty}^{\infty}\delta(t-nT)$	$\omega_0\sum_{k=-\infty}^{\infty}\delta(\omega-k\omega_0)$	$f_0\sum_{k=-\infty}^{\infty}\delta(f-kf_0)$	Sampling function
20.	$e^{-t^2/2\sigma^2}$	$\sigma\sqrt{2\pi}e^{-\sigma^2\omega^2/2}$	$\sigma\sqrt{2\pi}e^{-\sigma^2(2\pi f)^2/2}$	Gaussian signal

Table-2.2