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Control Systems

Electrical Engineering

Comprehensive Theory *with* Solved Examples

Civil Services Examination

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Corporate Office: 44-A/4, Kalu Sarai (Near Hauz Khas Metro Station), New Delhi-110016
E-mail: infomep@madeeasy.in
Contact: 9021300500
Visit us at: www.madeeasypublications.org

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Contents

Control Systems

Chapter 7
Time Domain Analysis of Control Systems 127

Chapter 8
Stability Analysis of Linear Control Systems 181

Chapter 9
The Root Locus Technique 195

Chapter 10
Frequency Domain Analysis of Control Systems 233

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CHAPTER 2

Transfer Function

2.1 Transfer Function and Impulse Response Function

In control systems, transfer functions are commonly used to characterise the input-output relationships of components or systems that can be described by linear, time-invariant differential equations.

Transfer Function

Consider the linear time invariant system defined by the following differential equation:

$$a_0 y^n + a_1 y^{n-1} + \ldots + a_{n-1} y^1 + a_n y = b_0 x^m + b_1 x^{m-1} + \ldots + b_{m-1} x^1 + b_m x$$

where y is output of system and x is input, $n > m$

The transfer function is defined as,

$$G(s) = \left.\frac{L(\text{output})}{L(\text{input})}\right|_{\text{Initial conditions are zero.}}$$

$$\therefore \qquad G(s) = \frac{Y(s)}{X(s)} = \frac{b_0 s^m + b_1 s^{m-1} + \ldots + b_{m-1} s + b_m}{a_0 s^n + a_1 s^{n-1} + \ldots + a_{n-1} s + a_n}$$

- The transfer function of a linear, time-invariant, differential equation system is defined as the ratio of the Laplace transform of the output (response function) to the Laplace transform of the input (driving function) under the assumption that all initial conditions are zero.
- The definition of transfer function is easily extended to a system with multiple inputs and outputs (i.e. a multivariable system). In a multivariable system, a linear differential equation may be used to describe the relationship between a pair of input and output variables, when all other inputs are set to zero. Since the principle of superposition is valid for linear systems, the total effect (on any output) due to all the inputs acting simultaneously is obtained by adding up the outputs due to each input acting alone.

2.2 Poles and Zeros of a Transfer Function

The transfer function of a linear control system can be expressed as

$$G(s) = \frac{A(s)}{B(s)} = \frac{K(s - s_1)(s - s_2)\ldots(s - s_n)}{(s - s_a)(s - s_b)\ldots(s - s_m)}$$

where K is known as gain factor of the transfer function $G(s)$.

In the transfer function expression, if s is put equal to s_a, s_b ... s_m then it is noted that the value of the transfer function is infinite. These s_a, s_b, ... s_m are called the ***poles*** of the transfer function.

In the transfer function expression, if s is put equal to s_1, s_2 ... s_n then it is noted that the value of the transfer function is zero. These s_1, s_2 ... s_n are called the ***zeros*** of the transfer function.

Multiple Poles and Multiple Zeros

The poles s_a, s_b ... s_m and the zeros s_1, s_2 ... s_n are either real or complex and the complex poles or zeros always appear in conjugate pairs.

It is possible that either poles or zeros may coincide; such poles or zeros are called ***multiple poles*** or ***multiple zeros***.

Simple Poles and Simple Zeros

Non-coinciding poles or zeros are called ***simple poles*** or ***simple zeros***. From the transfer function expression, it is observed that

- If $n > m$, then the value of transfer function is found to be infinity for $s = \infty$, Hence, it is concluded that there exists a pole of the transfer function at infinity (∞) and the multiplicity (order) of such a pole being $(n - m)$.
- If $n < m$, then the value of transfer function is found to be zero for $s = \infty$. Hence, it is concluded that there exists a zero of the transfer function at infinity (∞) and the multiplicity (order) of such a zero being $(m - n)$.

  Therefore, for a rational transfer function the total number of zeros is equal to the total number of poles.

  The transfer function of a system is completely specified in terms of its poles, zeros and the gain factor.

Consider the following transfer function:

$$G(s) = \frac{s+3}{(s+2)(s+1+3j)(s+1-3j)}$$

For the above transfer function, the poles are at

(a) $s_a = -2$ (b) $s_b = -1 - 3j$ and (c) $s_c = -1 + 3j$

The zeros are at $s_1 = -3$.

As the number of zeros should be equal to number of poles, the remaining two zeros are located at $s = \infty$. The pole-zero plot is plotted as shown:

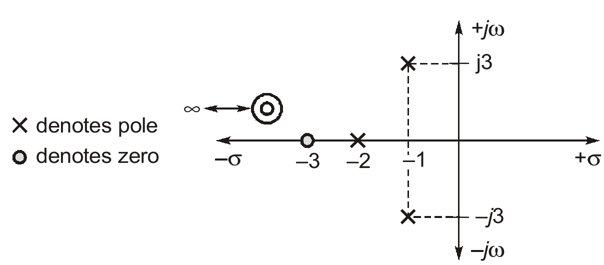


Fig. 2.1 : *Pole-zero plot*

Poles and zero are those complex/critical frequencies which make the transfer function infinity or zero.

Proper Transfer Functions

The transfer functions are said to be ***strictly proper*** if the order of the denominator polynomial is greater than that of the numerator polynomial (i.e. $m > n$). If $m = n$, the transfer function is called ***proper***. The transfer function is ***improper*** if $n > m$. Initial value theorem cannot be applied in such cases.

In the transfer function expression of a control system, the highest power of s in the numerator is generally either equal to or less than that of the denominator.

Sinusoidal Transfer Functions

The steady state response of a control system to a sinusoidal input is obtained by sinusoidal transfer function, which is arrived by replacing s with $j\omega$ in the transfer function of the system.

If the transfer function of a system is,

$$T(s) = \frac{C(s)}{R(s)} = \frac{K(s+s_1)(s+s_2)....(s+s_n)}{(s+s_a)(s+s_b).....(s+s_m)}$$

Then, sinusoidal (steady state) response is obtained by sinusoidal transfer function to a sinusoidal excitation,

$$T(j\omega) = \frac{C(j\omega)}{R(j\omega)} = \frac{K(j\omega+s_1)(j\omega+s_2).....(j\omega+s_n)}{(j\omega+s_a)(j\omega+s_b).....(j\omega+s_m)} = |T(j\omega)|\angle T(j\omega)$$

If the sinusoidal input $A\sin(\omega_0 t)$ is applied to the LTI system with transfer function $T(j\omega)$, then the output of the system $c(t)$ is given as,

$$c(t) = A|T(j\omega_0)|\cdot\sin(\omega_0 t + \angle T(j\omega_0))$$

2.3 Properties of Transfer Function

Properties of the transfer function are summarized as follows:

1. The concept of transfer function is applicable to the linear, time-invariant systems only.
2. The transfer function of a system is defined as the Laplace transform of its impulse response. Alternatively, it is the ratio of the Laplace transform of the output to the Laplace transform of the input.
3. All initial conditions of the system are set to zero to calculate transfer function.
4. The transfer function is a property of a system itself, thus independent of the magnitude and nature of the input variable or driving function.
5. The transfer function includes the information necessary to relate the input to the output, however, it does not provide any information concerning the physical structure of the system. The transfer functions of many physically different systems can be identical.
6. The transfer function of a continuous data system is expressed as a function of the complex variable(s) only. It is not a function of time, real variable or any other independent variable. Similarly, for discrete data system modelled by difference equations, the transfer function is expressed as a function of z.

Minimum and Non-minimum Phase Transfer Function

Transfer functions which have all poles and zeros in the left half of the s-plane i.e. system having no poles and zeros in the RHS of the s-plane are ***minimum phase transfer functions***. On the otherhand, a transfer function which has one or more zeros in the right half of s-plane is known as "***non-minimum phase transfer function***".

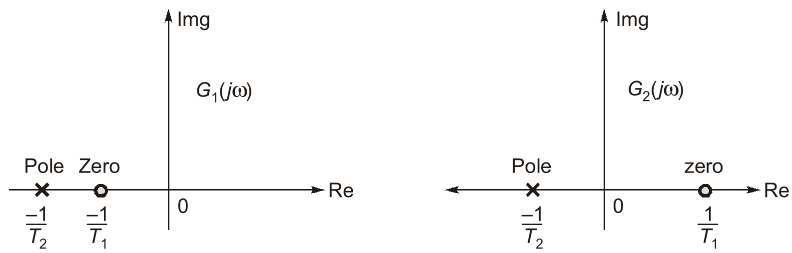
Let, $$G_1(s) = \frac{1+sT_1}{1+sT_2}; \quad 0 < T_2 < T_1$$

and $$G_2(s) = \frac{1-sT_1}{1+sT_2}; \quad 0 < T_2 < T_1$$

$\Rightarrow$ $$G_1(j\omega) = \frac{1+j\omega T_1}{1+j\omega T_2} \quad ...(i)$$

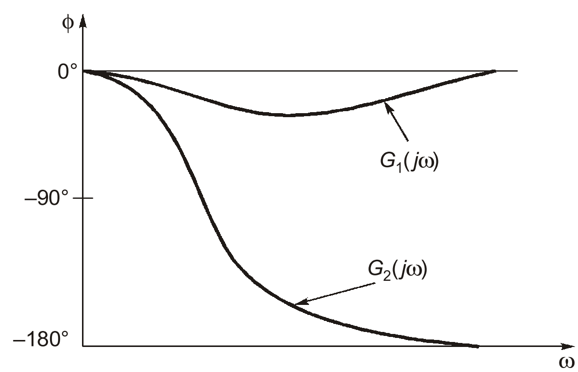
and $$G_2(j\omega) = \frac{1-j\omega T_1}{1+j\omega T_2} \quad ...(ii)$$

The transfer function given by equation (i) represents the minimum-phase transfer function and equation (ii) represents the non-minimum phase transfer function . The pole-zero configuration of above transfer function as given by equation (i) and (ii) may be drawn as:



Minimum phase transfer functions *Non-minimum phase transfer functions*

The ***minimum phase function*** has unique relationship between its phase and magnitude curves. Typical phase angle characteristics are shown below:



It will be seen that larger the phase lags present in a system, the more complex are its stabilization problems. Therefore in control systems, elements with non minimum phase transfer function are avoided as far as possible. A common example of a non-minimum phase system is "***transportation lag***" which has the transfer function,

$$G(j\omega) = e^{-j\omega T} = 1\angle{-\omega T} \text{ Radian}$$
$$= 1\angle{-57.3\,\omega T} \text{ degree}$$

2.4 Applicability of Transfer Functions

- It can be used for solving differential equations and for system analysis described by those differential equations.
- Transfer function describes the input-output behaviour of the system and does not deal with any information concerning the internal structure of the system. Functional operation of a system can be more readily visualized by examination of a block diagram rather than by the exhaustive analysis of the equations describing the physical system. Therefore, for a linear time-invariant system one can think of a system or its subsystems simply as interconnected blocks with each block described by a transfer function. The analysis of such systems can easily be done by transfer function approach.

Advantages of Transfer Function Approach

1. It gives simple mathematical algebraic equation.
2. It gives poles and zeros of the system directly.
3. Stability of the system can be determined easily.
4. The output of the system for any input can be determined easily.

5. If the transfer function of a system is known, the output or response can be studied for various forms of inputs.
6. The stability of a time-invariant linear system can be determined from the denominator polynomial of its transfer function, which is called characteristic equation of the system when equates to zero. The system is unstable if any roots of the characteristic equation (poles of the transfer function) lies in the right-hand side of s-plane.

Disadvantages of Transfer Function Approach

1. It is applicable only for LTI systems.
2. It does not take initial conditions into account.
3. The internal states of the system can not be determined.
4. Analysis of multiple input multiple output systems is cumbersome.
5. Controllability and observability can not be determined.

2.5 Transfer Functions of Control Loop Configurations

(a) **Closed loop control system:** In this configuration, the changes in the output are measured through feedback and compared with input to achieve the control objective.

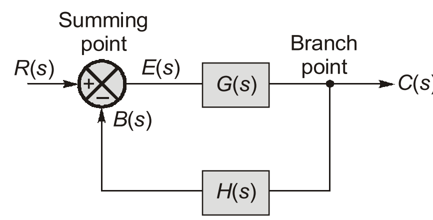


Fig. 2.2 : *Closed-loop control system*

$$E(s) = R(s) - B(s)$$
$$B(s) = C(s)\,H(s)$$
$$C(s) = E(s)\,G(s)$$

$$\therefore \quad \frac{C(s)}{G(s)} = R(s) - C(s)\,H(s)$$

$$\Rightarrow \quad C(s)\,[1 + G(s)\,H(s)] = G(s)\,R(s)$$

$$\Rightarrow \quad \frac{C(s)}{R(s)} = \frac{G(s)}{1 + G(s)\,H(s)}$$

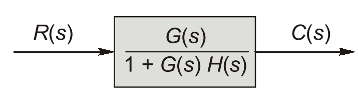


Fig. 2.3 : *Equivalent closed-loop system*

Hence, closed loop transfer function [C.L.T.F.] = $T(s) = \dfrac{G(s)}{1 + G(s)\,H(s)}$

For unity feedback system,

$$H(s) = 1$$

(b) **Open loop control system:** In this configuration, feedback is disconnected.

Transfer function of open loop system = $G(s)$

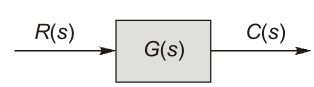


Fig. 2.4 : *Open-loop control system*

2.6 Interacting & Non-Interacting Systems

Interacting Systems

Consider the figure,

Transfer function, $$\frac{E_0(s)}{E_i(s)} = \frac{1/sC}{R + \dfrac{1}{sC}} = \frac{1}{sCR + 1} = \frac{1}{(s\tau + 1)}$$

where, $\tau = RC$ = Time constant

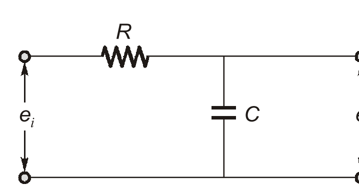


Fig. 2.5 : *RC circuit*

Consider the figure,
Assuming zero initial conditions,
In s-domain
Using KVL in loop-1

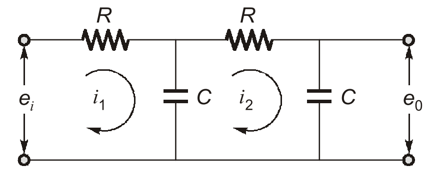


Fig. 2.6 : *RC circuits in cascade*

$$R\,I_1(s)+\frac{1}{sC}[I_1(s)-I_2(s)] = E_i(s)$$

Using KVL in loop-2

$$R\,I_2(s)+\frac{1}{sC}[I_2(s)-I_1(s)] = -\frac{1}{sC}[I_2(s)] = -E_0(s)$$

By eliminating $I_1(s)$ and $I_2(s)$ from the above equations, transfer function can be obtained as

$$\frac{E_0(s)}{E_i(s)} = \frac{1}{s^2R^2C^2+3sRC+1} = \frac{1}{s^2\tau^2+3s\tau+1}$$

where, $\tau = RC$ = time constant

Here, the transfer function of each of the individual *RC* circuits is $\frac{1}{(s\tau+1)}$. But, it is seen that overall transfer function of the two RC circuits connected in cascade is not equal to the product of two i.e. $\left[\frac{1}{(s\tau+1)}\cdot\frac{1}{(s\tau+1)}\right]$ but instead it is $\frac{1}{s^2\tau^2+3s\tau+1}$.

This difference is explained by the fact that while deriving the transfer function of single *RC* circuits, it is assumed that the output is unloaded. However, when the input of second circuit is obtained from the output of first, a certain amount of energy is drawn by the first circuit and hence its original transfer function is no longer valid. The degree to which the overall transfer function is modified from the product of individual transfer functions depends upon the amount of loading.

Hence, it can be concluded that when two time constant elements are cascaded interactively, the overall transfer function of such arrangement is not the product of two individual transfer functions, due to loading effects.

Non-Interacting Systems

Now consider, the system below having a subsystem block of constant gain *K* inserted between two R-C networks.

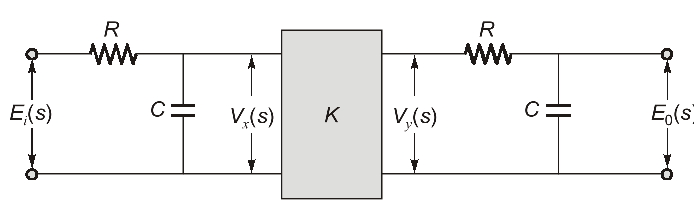


Fig. 2.7 : *RC circuited non-interacting*

$$\frac{V_x(s)}{E_i(s)} = \frac{1}{sRC+1}\ ;\ \frac{V_y(s)}{V_x(s)} = K\ ;\ \frac{E_0(s)}{V_y(s)} = \frac{1}{sRC+1}$$

$$\text{Overall transfer function} = \frac{E_o(s)}{E_i(s)} = \frac{K}{(sRC+1)^2}$$

So, when two systems are connected non-interactively, the overall transfer function of such arrangement is product of two individual transfer functions, due to absence of loading effect.

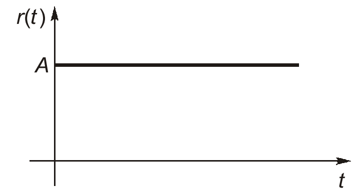
2.7 Standard Test Signals

1. Step Signal

$$r(t) = A\,u(t)$$

where, $$u(t) = \begin{cases} 1, & t > 0 \\ 0, & t < 0 \end{cases}$$

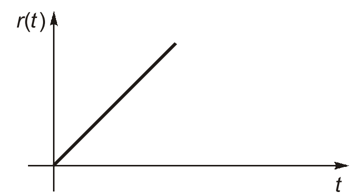
Laplace transform, $$R(s) = A/s$$



2. Ramp Signal

$$r(t) = \begin{cases} A\,t & t > 0 \\ 0 & t < 0 \end{cases}$$

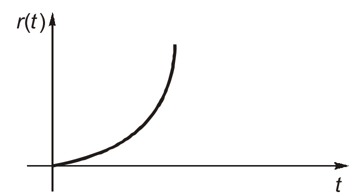
Laplace transform, $$R(s) = A/s^2$$



3. Parabolic Signal

$$r(t) = \begin{cases} A\,t^2/2, & t > 0 \\ 0\quad, & t < 0 \end{cases}$$

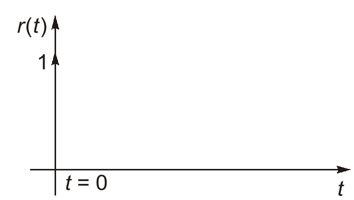
Laplace transform, $$R(s) = A/s^3$$



4. Impulse Signal

$$r(t) = \begin{cases} 0\;, & t \neq 0 \\ \int_{-\infty}^{\infty} \delta(t)\,dt = 1 \end{cases}$$

Laplace transform, $$R(s) = 1$$



Transfer function, $$G(s) = \frac{C(s)}{R(s)}$$

$$C(s) = G(s)\,R(s)$$

Let, $$R(s) = \text{Impulse signal} = 1$$

$$C(s) = \text{Impulse response} = G(s) \times 1 = \text{T.F.}$$

$$\mathcal{L}\{\text{Impulse Response}\} = \text{Transfer function} = \left[\frac{C(s)}{R(s)}\right]$$

The transfer function of an LTI system is equal to the Laplace transform of the impulse response of the system.

NOTE

- d/dt (Parabolic Response) = Ramp Response
- d/dt (Ramp Response) = Step Response
- d/dt (Step Response) = Impulse Response

2.8 Impulse Response of Linear Systems

By definition, the impulse response of a linear system is the output response of the system when the input is a unit impulse. For a single input single (SISO) system, if $r(t) = \delta(t)$, then transfer is,

$$G(s) = \frac{C(s)}{R(s)}$$

But, $R(s) = L[\delta(t)] = 1$

$\therefore$ $G(s) = C(s)$

Taking inverse Laplace transform on both sides m

$$c(t) = g(t)$$

where, $g(t)$ is the impulse response of the system.

Thus, Laplace transform of impulse response gives transfer function.

For a MIMO system, an impulse response matrix must be defined as,

$$g(t) = L^{-1}[G(s)]$$

In time domain it can be shown that,

$$c(t) = g(t) * r(t)$$

or, $$c(t) = \int_{-\infty}^{\infty} r(t)\, g(t-\tau)\, d\tau$$

$c(t) \rightarrow$ output in time domain.

$r(t) \rightarrow$ input signal in time domain.

2.9 Analogous Systems

In the analysis of linear systems, mathematical procedure for obtaining the solutions to a given set of equations does not depend upon what physical system the equations represent. Therefore, if the response of one physical system to a given excitation is determined, the responses of all other systems which can be described by the same set of equations are known for the similar excitation function. Systems which are governed by the same types of equations are called analogous systems. Thus on comparing equations (i), (ii), (iii) and (iv), following analogy can be developed:

Mechanical		Electrical	
Translation	Rotation	Force-voltage analogy	Force-current analogy
F	T	V	I
M	J	L	C
f	f	R	$1/R$
k	k	$1/C$	$1/L$
x	θ	Q	ϕ
v	ω	i	V

Table

NOTE: From the two analogies it is clear that mass and spring elements are energy storage elements.

2.10 Transfer Function of Mechanical Systems

There are multiple ways to arrive to the transfer function of complex mechanical systems, which may be classified as follows:

Direct Method

1. Mention the excitation and response (s) clearly in the system. Mark all the nodes/junctions.
2. Write down the force balance equations (D'Alembert's principle) at each node.
3. Convert all equations in Laplace domain, with all initial conditions as zero.
4. Rearrange the variables in the equations to arrive to the input-output equation.
5. Output (response) to input (excitation) ratio is the desired transfer function.

Indirect Method

1. Draw the equivalent diagram by force-voltage (series) analogy or force-current (parallel) analogy.
2. Write down the KVL and/or KCL equations.
3. Take Laplace transform with zero initial conditions.
4. Rewrite the input (excitation) output (response) equation by substitution.
5. Find the transfer function.

Nodal Method

In this method we draw the nodal diagram of the mechanical system keeping following point into account:

1. Number of principle nodes or nodes = Number of displacements.
2. Take an additional node which is a reference node (shows static point on earth).
3. Connect mass (or inertial mass) elements always between the principle node and reference node.
4. Connect other elements between the principle nodes or between principle nodes and reference depending on their position.
5. Thus obtain the nodal diagram and write down the describing (differential) equations at each node.

Finally transfer function may also be deduced by transferring all equations in Laplace domain with zero initial conditions.

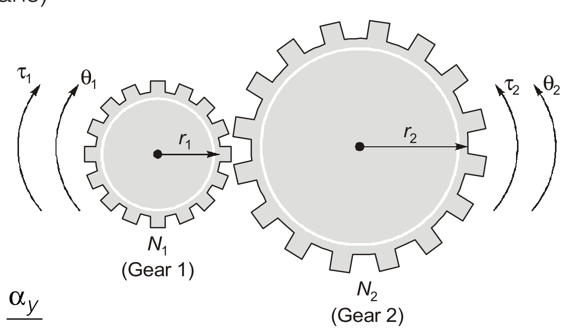
2.11 Gear Train

A gear train is a mechanical device which transfers energy from one part of the system to other part without any loss (ideally).

N = number of teeth on the circumference of gear wheel
r = radius of the gear wheel (m)
τ = torque (N-m)
θ = angular displacement (radians)

For two gear wheels: $$\frac{N_1}{N_2} = \frac{\tau_1}{\tau_2} = \frac{r_1}{r_2} = \frac{\theta_2}{\theta_1} = \frac{\dot{\theta}_2}{\dot{\theta}_1} = \frac{\ddot{\theta}_2}{\ddot{\theta}_1}$$

$$\therefore \quad \boxed{\frac{N_1}{N_2} = \frac{\tau_1}{\tau_2} = \frac{\theta_2}{\theta_1} = \frac{\omega_2}{\omega_1} = \frac{\alpha_2}{\alpha_1}}$$



Gear system

Similarly, if N gear wheels are cascaded,

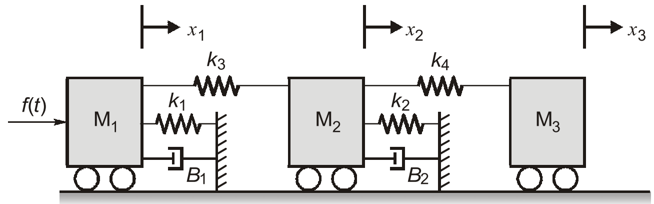
$$\frac{N_x}{N_y} = \frac{\tau_x}{\tau_y} = \frac{r_x}{r_y} = \frac{\theta_y}{\theta_x} = \frac{\omega_y}{\omega_x} = \frac{\alpha_y}{\alpha_x}$$

where, output is taken at y^{th} gear and input applied at x^{th} gear wheel.

Purpose of Gears

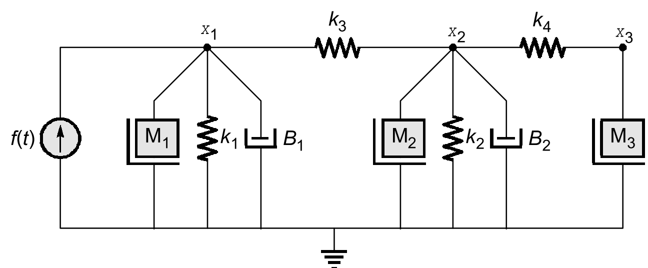
- Gears are mechanical structure, used as intermediate element between motors (shaft) and load.
- Gears are used for stepping up or stepping down either torque or speed.
- Gears are analogous to electrical transformers. Thus gear ratio is analogous to turn ratio or transformer ratio.

Example - 2.1 Draw the mechanical network for the system in figure given below and draw its analogous electrical circuit.



Solution:

The network diagram for the above system is shown in figure below:



Equation for node 'x_1':

$$f(t) = M_1\ddot{x}_1 + B_1\dot{x}_1 + k_1 x_1 + k_3 x_1 - k_3 x_2$$

Equation for node 'x_2':

$$k_3 x_1 - k_3 x_2 = M_2\ddot{x}_2 + B_2\dot{x}_2 + k_2 x_2 + k_4 x_2 - k_4 x_3$$

or, $M_2\ddot{x}_2 + B_2\dot{x}_2 + (k_2 + k_3 + k_4)x_2 - k_3 x_1 - k_4 x_3 = 0$

Equation for node 'x_3':

$$M_3\ddot{x}_3 = k_4 x_2 - k_4 x_3$$

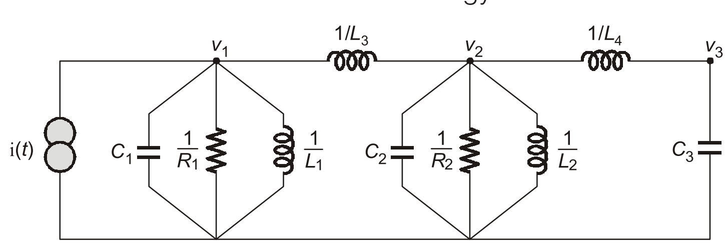
or, $M_3\ddot{x}_3 + k_4 x_3 - k_4 x_2 = 0$

Using force-current analogy, $i(t) = C_1\dfrac{dv_1}{dt} + \dfrac{v_1}{R_1} + \dfrac{1}{L_1}\int v_1\,dt + \dfrac{1}{L_3}\int v_1\,dt - \dfrac{1}{L_3}\int v_2\,dt$

$$C_2\frac{dv_2}{dt} + \frac{v_2}{R_2} + \left(\frac{1}{L_2} + \frac{1}{L_3} + \frac{1}{L_4}\right)\int v_2\,dt - \frac{1}{L_3}\int v_1\,dt - \frac{1}{L_4}\int v_3\,dt = 0$$

and, $C_3\dfrac{dv_3}{dt} + \dfrac{1}{L_4}\int v_3\,dt - \dfrac{1}{L_4}\int v_2\,dt = 0$

The analogous circuit based on force-current analogy is shown below:



where, $i(t) = f(t),\ C_1 = M_1,\ C_2 = M_2,\ C_3 = M_3,\ k_1 = \dfrac{1}{L_1}, k_2 = \dfrac{1}{L_2}, k_3 = \dfrac{1}{L_3} = k_{L_1} = \dfrac{1}{L_4}$

$$B_1 = \frac{1}{R_1} \text{ and } B_2 = \frac{1}{R_2}$$

Example - 2.2 Consider a satellite altitude control system with yaw angle θ. Let each jet thrust be $F/2$ so that a torque $T = fl$ is applied to the system. Let J be the moment of inertia about the axis of rotation at the centre of mass. Determine the transfer function.

Solution:

Step-1: Write differential equation governing the system.

Step-2: Take Laplace transform of the differential equation assuming all initial conditions to be zero.

Step-3: Take the ratio of output $\theta(s)$ to input $T(s)$.

Applying Newton's second law, the differential equation governing the system is,

$$J\frac{d^2\theta}{dt^2} = T$$

Taking Laplace transform on both sides,

$$Js^2\,\theta(s) = T(s)$$

$\therefore$ Transfer function is given by,

$$\frac{\theta(s)}{T(s)} = \frac{1}{Js^2}$$

Example - 2.3 Derive the transfer function of a turbo-propeller engine having multiple input variables (fuel rate and propeller blade angle) and multiple output variables (speed of engine rotation and turbine temperature).

Solution:

The following equations may be considered for the given MIMO system:

$$C_1(s) = G_{11}(s)\,R_1(s) + G_{12}(s)\,R_2(s)$$

$$C_2(s) = G_{21}(s)\,R_1(s) + G_{22}(s)\,R_2(s)$$

where, $C_1(s)$ = Transformed variable of speed of rotation

$C_2(s)$ = Transformed variable of turbine inlet temperature

$R_1(s)$ = Transformed variable of fuel rate

$R_2(s)$ = Transformed variable of propeller blade angle

Since given system is assumed linear, superposition principle holds:

$\therefore$ $G_{11}(s)$ = Transfer function between fuel rate and speed of rotation of engine with propeller blade angle held at reference value (i.e. $R_2(s) = 0$)

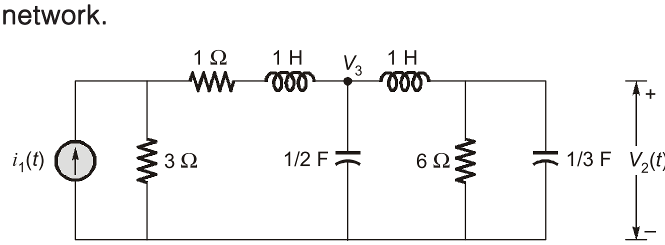
Similarly, we can define $G_{12}(s)$, $G_{21}(s)$ and $G_{22}(s)$.

In matrix form, $C(s) = G(s)\,R(s)$

$$\begin{bmatrix} C_1(s) \\ C_2(s) \end{bmatrix} = \begin{bmatrix} G_{11}(s) & G_{12}(s) \\ G_{21}(s) & G_{22}(s) \end{bmatrix}\begin{bmatrix} R_1(s) \\ R_2(s) \end{bmatrix}$$

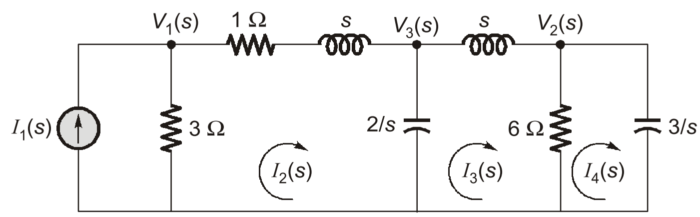
where, $G(s)$ represents the transfer function matrix.

Example - 2.4 Transfer function electrical network with current source. Obtain the transfer function of the following network.



Solution:

Transforming the given network in s-domain and writing equations for node voltages and branch currents.



$$V_1 = (I_1 - I_2)\times 3 \qquad I_2 = \frac{V_1 - V_3}{s+1}$$

$$V_3 = (I_2 - I_3)\times\frac{2}{s} \qquad I_3 = \frac{V_3 - V_2}{s}$$

$$V_2 = (I_3 - I_4)\times 6 \qquad I_4 = V_2\frac{s}{3}$$

$$I_2 = \frac{V_1 - V_3 - I_2}{s}$$

$$I_1 = \frac{V_1}{3} + I_2 = \frac{1}{3}\times\{I_2(s+1)+V_3\} + I_2$$

$$= \frac{I_2}{3}(s+4) + \frac{V_3}{3}$$

$$= \frac{1}{3}\left\{\left(\frac{sV_3}{2}+I_3\right)(s+4)+V_3\right\} = \frac{1}{3}\left\{V_3\left(\frac{s^2}{2}+2s+1\right)+I_3(s+4)\right\}$$

$$= \frac{1}{3}\left\{(sI_3+V_2)\left(\frac{s^2}{2}+2s+1\right)+I_3(s+4)\right\}$$

$$= \frac{1}{3}\left\{I_3\left(\frac{s^3}{2}+2s^2+2s+4\right)+V_2\left(\frac{s^2}{2}+2s+1\right)\right\}$$

$$= \frac{1}{3}\left\{\left(\frac{V_2}{6}+I_4\right)\left(\frac{s^3}{2}+2s^2+2s+4\right)+V_2\left(\frac{s^2}{2}+2s+1\right)\right\}$$

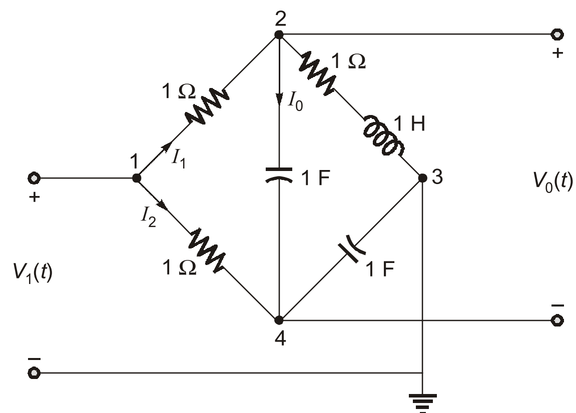
$$= \frac{1}{3}\left\{V_2\left(\frac{s^3}{12}+\frac{5}{6}s^2+\frac{7}{3}s+\frac{5}{3}\right)+I_4\left(\frac{s^3}{2}+2s^2+2s+4\right)\right\}$$

$$= \frac{1}{3}\left\{\frac{V_2}{12}(s^3+10s^2+28s+20)+\frac{V_2}{3}\left(\frac{s^4}{2}+2s^3+2s^2+4s\right)\right\}$$

$$= \frac{V_2}{36}(2s^4+9s^3+18s^2+44s+20)$$

$$\therefore \qquad \frac{V_2(s)}{I_1(s)} = \frac{36}{2s^4+9s^3+18s^2+44s+20}$$

Example - 2.5 Transfer function electrical network with voltage source.



Calculate $V_0(s)/V_1(s)$.

Solution:

The equation for branch currents and node voltages are:

$$I_1 = V_1 - V_2 \qquad V_2 = (I_1 - I_0)(s+1)$$

$$I_0 = (V_2 - V_4)s \qquad V_4 = (I_2 + I_0)\frac{1}{s}$$

$$I_2 = V_1 - V_4 \qquad V_0 = V_2 - V_4$$

$$V_1 = I_1 + V_2$$

$$= \frac{V_2}{s+1} + I_0 + V_2 = V_2\left(\frac{s+2}{s+1}\right) + I_0$$

$$= \left(\frac{I_0}{s} + V_4\right)\left(\frac{s+2}{s+1}\right) + I_0$$

$$V_1 = I_0\left(\frac{s^2+2s+2}{s^2+s}\right) + V_4\left(\frac{s+2}{s+1}\right)$$

$$V_4 = (I_2 + I_0)\times\frac{1}{s}$$

$\Rightarrow$
$$sV_4 = V_1 - V_4 + I_0 \text{ or } V_4 = \frac{V_1 + I_0}{s+1}$$

$\therefore$
$$V_1 = I_0\left(\frac{s^2+2s+2}{s^2+s}\right) + \frac{V_1+I_0}{s+1}\left(\frac{s+2}{s+1}\right)$$

$$V\left(1 - \frac{s+2}{(s+1)^2}\right) = \frac{I_0}{s}\left(\frac{s^3+4s^2+6s+2}{(s+1)^2}\right) = V_0\frac{s^3+4s^2+6s+2}{(s+1)^2}$$

$\therefore$
$$\frac{V_0(s)}{V_1(s)} = \frac{s^2+s-1}{s^3+4s^2+6s+2}$$