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Electromagnetic Theory

Electrical Engineering

Comprehensive Theory *with* Solved Examples

Civil Services Examination

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Electromagnetic Theory

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CHAPTER 1

Introduction of Electromagnetic Theory

1.1 Introduction

Electromagnetics is the study of the effects of electric charge at rest and in motion. There are two kind of charges: positive and negative. Both charges produce a current, which gives rise to a magnetic field. A field is spatial distribution of a quantity which may or may not be a function of time. A time varying electric field is accompanied by a magnetic field, and vice versa. Time varying electric and magnetic fields are coupled, producing electromagnetic field. Under certain conditions, time varying electromagnetic fields produce waves that radiates from the source.

Electromagnetic deals with space concepts and required thinking in three dimensions of real world; hence we must understand the three dimensional coordinate systems.

1.2 Coordinate Systems

In general, the physical qualities we shall be dealing with in electromagnetics are functions of space and time. In order to describe the spatial variations of the quantities, we must be able to define all points uniquely in space in a suitable manner. This requires using an appropriate coordinate system.

A coordinate system defines points of reference from which specific vector directions may be defined.

Depending on the geometry of the application, one coordinate system may lead to more efficient vector definitions than others. The three most commonly used co-ordinate systems used in the study of electromagnetics are rectangular coordinates (or cartesian coordinates), cylindrical coordinates and spherical coordinates.

1.2.1 Cartesian Coordinates

A vector $\vec{A}$ in Cartesian (other wise known as rectangular) coordinates can be written as

(A_x, A_y, A_z) or $A_x\hat{a}_x + A_y\hat{a}_y + A_z\hat{a}_z$ ...(1.1)

Where a_x, a_y, a_z are unit vectors along the x, y and z directions

The ranges of the variables are:

$-\infty \leq x \leq +\infty$

$-\infty \leq y \leq +\infty$

$-\infty \leq z \leq +\infty$ ...(1.2)

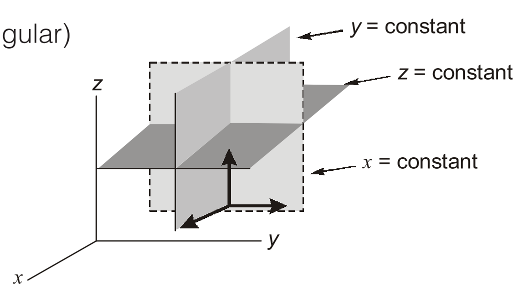


***Figure 1.1 :** A point in Cartesian coordinates is defined by the intersection of the three planes: x = constant, y = constant, z = constant. The three unit vectors are normal to each of the three surfaces.*

1.2.2 Cylindrical Coordinates

The cylindrical coordinate system is very convenient whenever we are dealing with problems having cylindrical symmetry.

A point *P* in cylindrical coordinates is represented as (ρ, ϕ, z) and is as shown in Fig 1.2. Observe Fig. 1.2 closely and note how we define each space variable; ρ is the radius of the cylinder passing through *P* or the radial distance from the *z*-axis; ϕ, called the azimuthal angle, is measured from the *x*-axis in the *xy*-plane; and *z* is the same as in the Cartesian system. The ranges of the variables are:

$$0 \le \rho \le \infty \qquad ...(1.3)$$
$$0 \le \phi \le 2\pi$$
$$-\infty \le z \le +\infty$$

A vector $\vec{A}$ in cylindrical coordinates can be written as

$$(A_\rho, A_\phi, A_z) \text{ or } A_\rho \hat{a}_\rho + A_\phi \hat{a}_\phi + A_z \hat{a}_z \qquad ...(1.4)$$

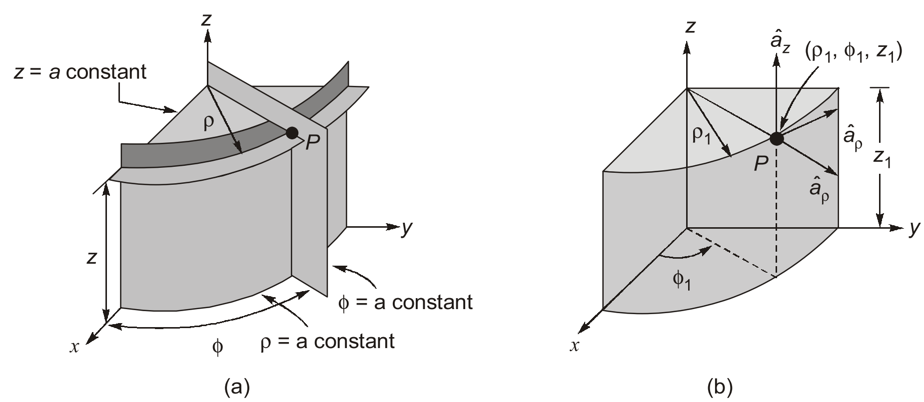


Figure 1.2: *(a) The point is defined by the intersection of the cylinder and the two planes.*
(b) Point P and unit vectors in the cylindrical coordinate system.

Notice that the unit vectors $\hat{a}_\rho, \hat{a}_\phi$ and $\hat{a}_z$ are mutually perpendicular because our coordinate system is orthogonal.

$$\hat{a}_\rho \cdot \hat{a}_\phi = \hat{a}_\phi \cdot \hat{a}_z = \hat{a}_z \cdot \hat{a}_\rho = 0$$
$$\hat{a}_\rho \cdot \hat{a}_\rho = \hat{a}_\phi \cdot a_\phi = \hat{a}_z \cdot \hat{a}_z = 1 \qquad ...(1.5)$$
$$\hat{a}_\rho \times \hat{a}_\phi = \hat{a}_z$$
$$\hat{a}_\phi \times \hat{a}_z = \hat{a}_\rho$$
$$\hat{a}_z \times \hat{a}_\rho = \hat{a}_\phi \qquad ...(1.6)$$

Note: An orthogonal system is one in which the coordinates are mutually perpendicular.

The relationships between the variables (x, y, z) of the Cartesian coordinate system and those of the cylindrical system (ρ, ϕ, z) are easily obtained from figure 1.3.

Point transformation, $\rho = \sqrt{x^2 + y^2}, \ \phi = \tan^{-1}\frac{y}{x}, \ z = z \qquad ...(1.7)$

or,

$$x = \rho \cos\phi, \ y = \rho \sin\phi, \ z = z \qquad ...(1.8)$$

Whereas equation (1.7) is for transforming a point from Cartesian (x, y, z) to cylindrical (ρ, ϕ, z) coordinates, equation (1.8) is for $(\rho, \phi, z) \to (x, y, z)$ transformation.

The relationships between $\hat{a}_x, \hat{a}_y, \hat{a}_z$ and $\hat{a}_\rho, \hat{a}_\phi, \hat{a}_z$ are

Vector transformation, $\hat{a}_x = \cos\phi\hat{a}_\rho - \sin\phi\hat{a}_\phi$

$$\hat{a}_y = \sin\phi\hat{a}_\rho + \cos\phi\hat{a}_\phi$$

$$\hat{a}_z = \hat{a}_z \qquad ...(1.9)$$

or,

$$\hat{a}_\rho = \cos\phi\hat{a}_x + \sin\phi\hat{a}_y$$

$$\hat{a}_\phi = -\sin\phi\hat{a}_x + \cos\phi\hat{a}_y$$

$$\hat{a}_z = \hat{a}_z \qquad ...(1.10)$$

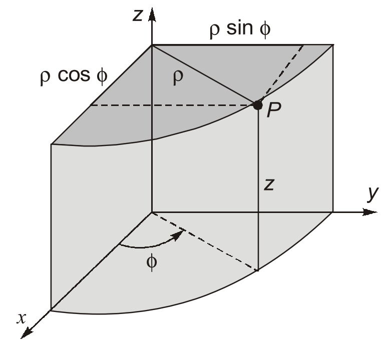


Figure 1.3: *Relationship between (x, y, z) and (ρ, ϕ, z)*

Finally, the relationship between (A_x, A_y, A_z) and (A_ρ, A_ϕ, A_z) are

$$\begin{vmatrix} A_\rho \\ A_\phi \\ A_z \end{vmatrix} = \begin{vmatrix} \cos\phi & \sin\phi & 0 \\ -\sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{vmatrix} \begin{vmatrix} A_x \\ A_y \\ A_z \end{vmatrix} \qquad ...(1.11)$$

$$\begin{vmatrix} A_x \\ A_y \\ A_z \end{vmatrix} = \begin{vmatrix} \cos\phi & -\sin\phi & 0 \\ \sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{vmatrix} \begin{vmatrix} A_\rho \\ A_\phi \\ A_z \end{vmatrix} \qquad ...(1.12)$$

1.2.3 Spherical Coordinates

The spherical coordinate system is most appropriate when dealing with problems having a degree of spherical symmetry. A point *P* can be represented as (r, θ, ϕ) and is illustrated in figure. 1.4.

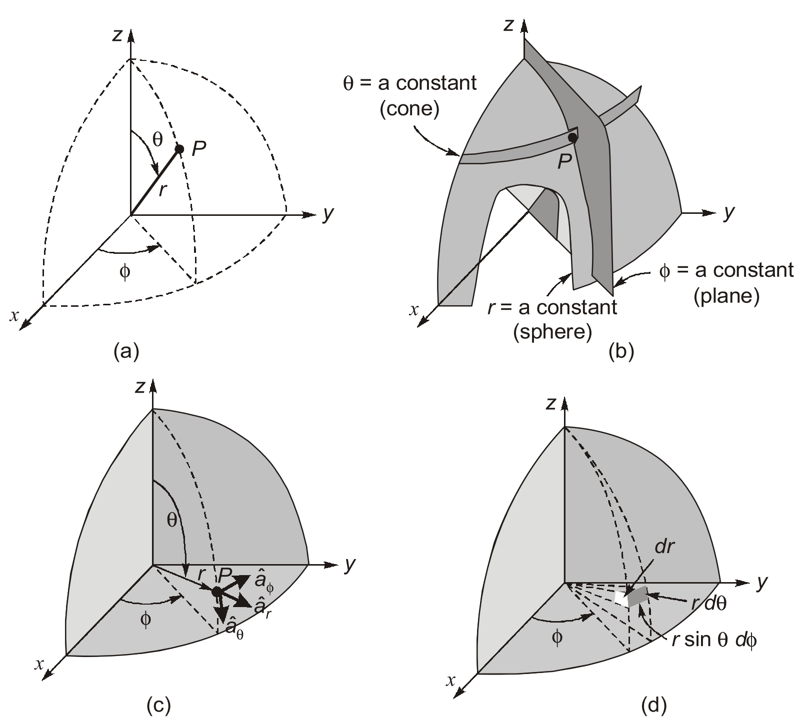


Figure 1.4 : *(a) Point P and unit vectors in the cylindrical coordinate system (b) The three mutually perpendicular surfaces of the spherical coordinate system (c) The three unit vectors of spherical coordinates (d) The differential volume element in the spherical coordinate system.*

From figure. 1.4, we notice that r is defined as the distance from the origin to point P or the radius of a sphere centered at the origin and passing through P; θ (called the colatitudes) is the angle between the z-axis and the position vector of P; and ϕ is measured from the x-axis (the same azimuthal angle in cylindrical coordinates). According to these definitions, the ranges of the variables are

$$0 \le r \le \infty \quad ...(1.13)$$
$$0 \le \theta \le \pi$$
$$0 \le \phi \le 2\pi$$

A vector $\vec{A}$ in spherical coordinates can be written as

$$(A_r, A_\theta, A_\phi) \text{ or } A_r\hat{a}_r + A_\theta\hat{\phi}_\theta + A_\phi\hat{a}_\phi \quad ...(1.14)$$

The unit vectors $\hat{a}_r$, $\hat{a}_\theta$, and $\hat{a}_\phi$ are mutually perpendicular because our coordinate system is orthogonal.

$$\hat{a}_r \cdot \hat{a}_\theta = \hat{a}_\theta \cdot \hat{a}_\phi = \hat{a}_\phi \cdot a_r = 0$$
$$\hat{a}_r \cdot \hat{a}_r = \hat{a}_\theta \cdot a_\theta = \hat{a}_\phi \cdot \hat{a}_\phi = 1 \quad ...(1.15)$$
$$\hat{a}_r \times \hat{a}_\theta = \hat{a}_\phi$$
$$\hat{a}_\theta \times \hat{a}_\phi = \hat{a}_r$$
$$\hat{a}_\phi \times \hat{a}_r = \hat{a}_\theta \quad ...(1.16)$$

The relationship between the variables (x, y, z) of the Cartesian coordinate system and those of the spherical coordinate system (ρ, θ, ϕ) are easily obtained from figure 1.4.

Point transformation, $$r = \sqrt{x^2+y^2+z^2},\ \theta = \tan^{-1}\frac{\sqrt{x^2+y^2}}{z},\ \phi = \tan^{-1}\frac{y}{x} \quad ...(1.17)$$

or $$x = r\sin\theta\cos\phi,\ y = r\sin\theta\sin\phi,\ z = r\cos\theta \quad ...(1.18)$$

The relationship between $\hat{a}_x, \hat{a}_y, \hat{a}_z$ and $\hat{a}_r, \hat{a}_\theta, \hat{a}_\phi$ are

$$\hat{a}_x = \sin\theta\cos\phi\hat{a}_r + \cos\theta\cos\phi\hat{a}_\theta - \sin\phi\hat{a}_\phi$$
$$\hat{a}_y = \sin\theta\sin\phi\hat{a}_r + \cos\theta\sin\phi\hat{a}_\theta + \cos\phi\hat{a}_\phi$$
$$\hat{a}_z = \cos\phi\hat{a}_r - \sin\phi\hat{a}_\phi$$

or, $$\hat{a}_r = \sin\theta\cos\phi\hat{a}_x + \sin\theta\sin\phi\hat{a}_y + \cos\phi\hat{a}_z$$
$$\hat{a}_\theta = \cos\theta\cos\phi\hat{a}_x + \cos\theta\sin\phi\hat{a}_y - \sin\theta\hat{a}_z$$
$$\hat{a}_\phi = -\sin\phi\hat{a}_x + \cos\phi\hat{a}_y \quad ...(1.19)$$

Finally, the relationship between (A_x, A_y, A_z) and (A_r, A_θ, A_ϕ) are

Vector transformation, $$\begin{vmatrix} A_r \\ A_\theta \\ A_\phi \end{vmatrix} = \begin{vmatrix} \sin\theta\cos\phi & \sin\theta\sin\phi & \cos\theta \\ \cos\theta\cos\phi & \cos\theta\sin\phi & -\sin\theta \\ -\sin\phi & \cos\phi & 0 \end{vmatrix} \begin{vmatrix} A_x \\ A_y \\ A_z \end{vmatrix} \quad ...(1.20)$$

or, $$\begin{vmatrix} A_x \\ A_y \\ A_z \end{vmatrix} = \begin{vmatrix} \sin\theta\cos\phi & \cos\theta\cos\phi & -\sin\phi \\ \sin\theta\cos\phi & \cos\theta\sin\phi & \cos\phi \\ \cos\theta & -\sin\theta & 0 \end{vmatrix} \begin{vmatrix} A_r \\ A_\theta \\ A_\phi \end{vmatrix} \quad ...(1.21)$$

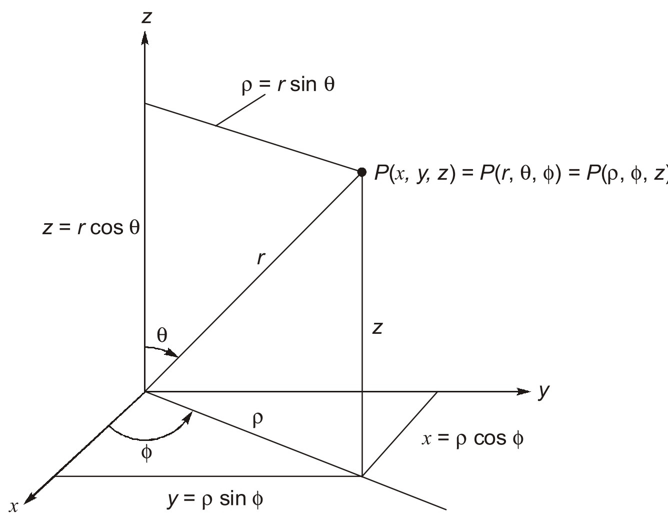


Figure 1.5: *Relationships between space variables (x, y, z), (r, θ, φ) and (ρ, φ, z)*

Example - 1.1 Find the component of a vector $\vec{A} = -z\hat{a}_y + y\hat{a}_z$ at the point $P(0, -2, 3)$ which si directed towards the point $Q(\sqrt{3}, -60°, 1)$.

Solution:

$P(0, -2, 3)$ is in Cartesian coordinates

$Q(\sqrt{3}, -60°, 1)$ is in cylindrical coordinates

$$x = r\cos\theta = \sqrt{3}\cos(-60°) = 0.866$$

$$y = r\sin\theta = \sqrt{3}\sin(-60°) = -1.5$$

$$Q = (0.866, -1.5, 1)$$

Vector
$$\vec{r}_{PQ} = (0.866 - 0)\,\hat{a}_x + (-1.5 + 2)\,\hat{a}_y + (1 - 3)\,\hat{a}_z$$
$$= 0.866\,\hat{a}_x + 0.5\,\hat{a}_y - 2\,\hat{a}_z$$

Unit vector
$$\hat{a}_{PQ} = \frac{0.866\hat{a}_x + 0.5\hat{a}_y - 2\hat{a}_z}{\sqrt{0.866^2 + 0.5^2 + 2^2}}$$

Component of vector $\vec{A}$ at point $P(0, -2, 3)$ towards point Q

$$\vec{A}\cdot\hat{a}_{PQ} = (-3\hat{a}_y - 2\hat{a}_z)\cdot\frac{(0.866\hat{a}_x + 0.5\hat{a}_y - 2\hat{a}_z)}{2.236}$$

$$= \frac{(-3\times 0.5) + 4}{2.236} = 1.118$$

Example - 1.2 Determine the curl of the following vector fields:

(i) $\vec{A} = \rho z^2\hat{a}_\rho + \rho\sin^2\phi\,\hat{a}_\phi + 2\rho z\sin^2\phi\,\hat{a}_z$, in circular cylindrical coordinate system.

(ii) $\vec{B} = r\,\hat{a}_r + r\cos^2\theta\,\hat{a}_\phi$, in spherical coordinate system.

Solution:

(i) For circular cylindrical coordinate system,

$$\nabla \times \vec{A} = \frac{1}{\rho}\begin{vmatrix} \hat{a}_\rho & \rho\hat{a}_\phi & \hat{a}_z \\ \frac{\partial}{\partial\rho} & \frac{\partial}{\partial\phi} & \frac{\partial}{\partial z} \\ A_\rho & \rho A_\phi & A_z \end{vmatrix} = \frac{1}{\rho}\begin{vmatrix} \hat{a}_\rho & \rho\hat{a}_\phi & \hat{a}_z \\ \frac{\partial}{\partial\rho} & \frac{\partial}{\partial\phi} & \frac{\partial}{\partial z} \\ \rho z^2 & \rho^2\sin^2\phi & 2\rho z\sin^2\phi \end{vmatrix}$$

$$= \frac{1}{\rho}\left[(2\rho z\sin 2\phi)\hat{a}_\rho - \rho(2z\sin^2\phi - 2\rho z)\hat{a}_\phi + (2\rho\sin^2\phi)\hat{a}_z\right]$$

$$= 2z\sin 2\phi\,\hat{a}_\rho + 2z(\rho - \sin^2\phi)\hat{a}_\phi + 2\sin^2\phi\,\hat{a}_z$$

(ii) For spherical coordinate system,

$$\nabla \times \vec{B} = \frac{1}{r^2\sin\theta}\begin{vmatrix} \hat{a}_r & r\hat{a}_\theta & r\sin\theta\,\hat{a}_\phi \\ \frac{\partial}{\partial r} & \frac{\partial}{\partial\theta} & \frac{\partial}{\partial\phi} \\ A_r & rA_\theta & r\sin\theta A_\phi \end{vmatrix} = \frac{1}{r^2\sin\theta}\begin{vmatrix} \hat{a}_r & r\hat{a}_\theta & r\sin\theta\,\hat{a}_\phi \\ \frac{\partial}{\partial r} & \frac{\partial}{\partial\theta} & \frac{\partial}{\partial\phi} \\ r & 0 & r^2\sin\theta\cos^2\theta \end{vmatrix}$$

$$= \frac{1}{r^2\sin\theta}\left[\frac{\partial}{\partial\theta}(r^2\sin\theta\cos^2\theta)\hat{a}_r - 2r^2\sin\theta\cos^2\theta\,\hat{a}_\theta\right]$$

$$= \frac{\cos^3\theta - 2\cos\theta\sin^2\theta}{\sin\theta}\hat{a}_r - 2\cos^2\theta\,\hat{a}_\theta$$

1.3 Vector Calculus

In electromagnetics, we frequently use the concept of a **field**. A field is a function that assigns a particular physical quantity to every point in a region. In general, a field varies with both position and time. There are scalar fields and vector fields.

Quantities that can be described by a magnitude alone are called **scalars**. Distance, temperature, mass etc. are examples of scalar quantities. Other quantities, called **vectors**, require both a magnitude and a direction to fully characterize them. Examples of vector quantities include velocity, force, acceleration etc.

The concepts introduced in this section provide a convenient language for expressing certain fundamental ideas in electromagnetics in general.

1.3.1 Line, Surface, and Volume Integrals

Line Integral

The familiar concept of integration will now be extended to cases when the integrand involves a vector. By a line we mean the path along a curve in space. We shall use terms such as line, curve, and contour interchangeably.

The line integral $\int_L \vec{A}.\vec{dl}$ is the integral of the tangential component of $\vec{A}$ along curve L.

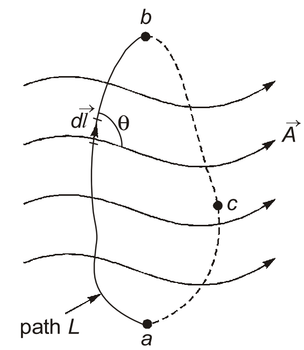


Figure 1.10 : *Path of integration of vector field A.*

Given a vector field $\vec{A}$ and a curve L, we define the integral as the line integral of $\vec{A}$ around L (see figure 1.10):

$$\int_L \vec{A} \cdot \vec{dl} = \int_a^b |\vec{A}| \cos\theta \, dl \quad ...(1.22)$$

If the path of integration is a closed curve such as abca in figure 1.11, precedent equation becomes a closed contour integral.

$$\oint_L \vec{A} \cdot \vec{dl} \quad ...(1.23)$$

Which is called the circulation of $\vec{A}$ around L.

Surface Integral

Another integral that will be encountered in the study of electromagnetic fields is the surface integral. Given a vector field $\vec{A}$, continuous in a region containing the smooth surface S, we define the surface integral or the flux of $\vec{A}$ through S (see figure 1.11)

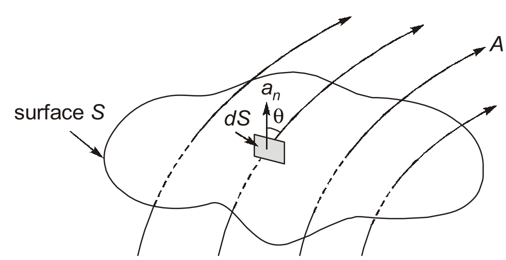


Figure 1.11: *The flux of a vector field $\vec{A}$ through surface S.*

$$\psi = \int_S |\vec{A}| \cos\theta \, dS = \int_S \vec{A} \cdot \hat{a}_n \, dS \quad ...(1.24)$$

Or simply

$$\psi = \int_S \vec{A} \cdot d\vec{S} \quad ...(1.25)$$

Where, at any point on S, $\hat{a}_n$ is the unit normal to S. For a closed surface (defining a volume), precedent equation becomes:

$$\psi = \oint_S \vec{A} \cdot d\vec{S} \quad ...(1.26)$$

Which is referred to as the net outward flux of $\vec{A}$ from S.

Volume Integral

Finally, we will encounter various volume integrals of scalar quantities, such as a volume charge density ρ_v. A typical integration would involve the computation of the total charge if the volume charge density was known. It is written as:

$$Q = \int_v \rho_v \, dv \quad ...(1.27)$$

1.3.2 Differential Length, Area and Volume

In our study of electromagnetism we will often be required to perform line, surface, and volume integrations. The evaluation of these integrals in a particular coordinate system requires the knowledge of differential elements of length, surface, and volume. In the following subsections we describe how these differential elements are constructed in each coordinate system.

Cartesian Coordinates

From figure 1.6, we notice that:

1. Differential displacement is given by:

$$\vec{dl} = dx\hat{a}_x + dy\hat{a}_y + dz\hat{a}_z \quad ...(1.28)$$

2. Differential normal area is given by:

$$\vec{dS} = \begin{cases} dydz\hat{a}_x \\ dxdz\hat{a}_y \\ dxdy\hat{a}_z \end{cases} \quad ...(1.29)$$

3. Differential volume is given by:

$$dv = dx\,dy\,dz \quad ...(1.30)$$

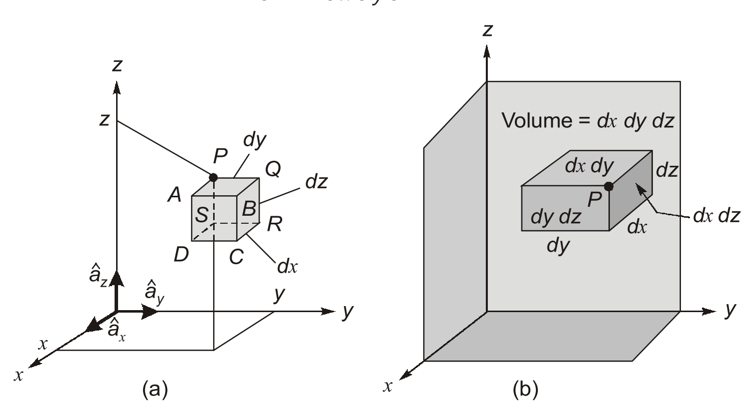


***Figure 1.6 :** Differential elements in the right-handed Cartesian coordinate system*

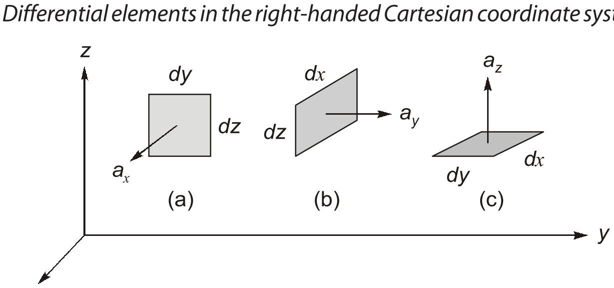


***Figure 1.7 :** Differential normal areas in Cartesian coordinates.*

The way $\vec{dS}$ is defined is important. The differential surface (or area) element $\vec{dS}$ may generally be defined as:

$$\vec{dS} = dS\hat{a}_n \quad ...(1.31)$$

where dS is the area of the surface element and $\hat{a}_n$ is a unit vector normal to the surface dS (and directed away) from the volume if dS is part of the surface describing a volume). If we consider surface *ABCD* in figure 1.6, for example, $\vec{dS} = dydz\hat{a}_x$ whereas for surface *PQRS*, $\vec{dS} = -dydz\hat{a}_x$ because $\hat{a}_n = -\hat{a}_x$ is normal to *PQRS*.

Remember: What we have to remember at all times about differential elements is $\vec{dl}$ and how to get $\vec{dS}$ and dv from it. Once dl is remembered, $\vec{dS}$ and dv can easily be found.

Cylindrical Coordinates:

From figure 1.8, we notice that:

1. Differential displacement is given by:

$$\vec{dl} = d\rho\hat{a}_\rho + \rho d\phi\hat{a}_\phi + dz\hat{a}_z \quad ...(1.32)$$

2. Differential normal area is given by:

$$\vec{dS} = \begin{cases} \rho d\phi dz\hat{a}_\rho \\ d\rho dz\hat{a}_\phi \\ \rho d\phi d\rho\hat{a}_z \end{cases} \quad ...(1.33)$$

3. Differential volume is given by:

$$dv = \rho d\rho d\phi dz \quad ...(1.34)$$

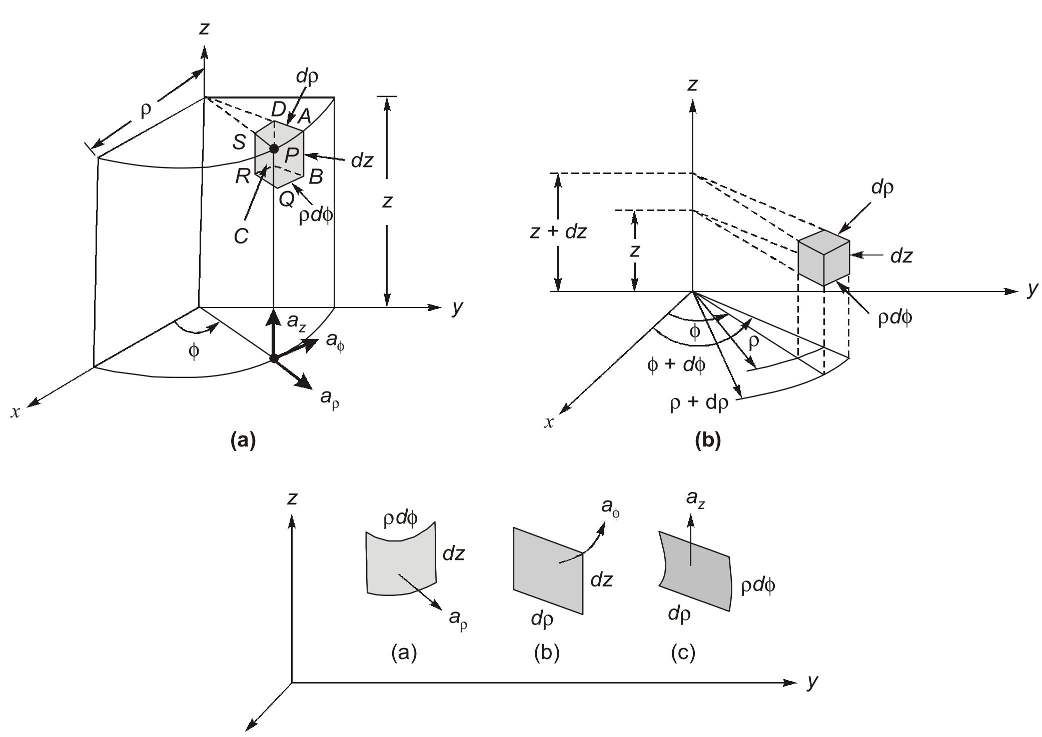


Figure 1.8 : *(a) & (b) Differential elements in cylindrical coordinates*
(c) Differential normal areas in cylindrical coordinates

Spherical Coordinates:

From figure 1.9, we notice that:

1. Differential displacement is given by:

$$\vec{dl} = dr\hat{a}_r + rd\theta\hat{a}_\theta + r\sin\theta d\phi\hat{a}_\phi \quad ...(1.35)$$

2. Differential normal area is given by:

$$\vec{dS} = \begin{cases} r^2\sin\theta d\theta d\phi\hat{a}_r \\ r\sin\theta dr d\phi\hat{a}_\theta \\ r\,dr\,d\theta\hat{a}_\phi \end{cases} \quad ...(1.36)$$

3. Differential volume is given by:

$$dv = r^2 \sin\theta\, dr\, d\theta\, d\phi \quad ...(1.37)$$

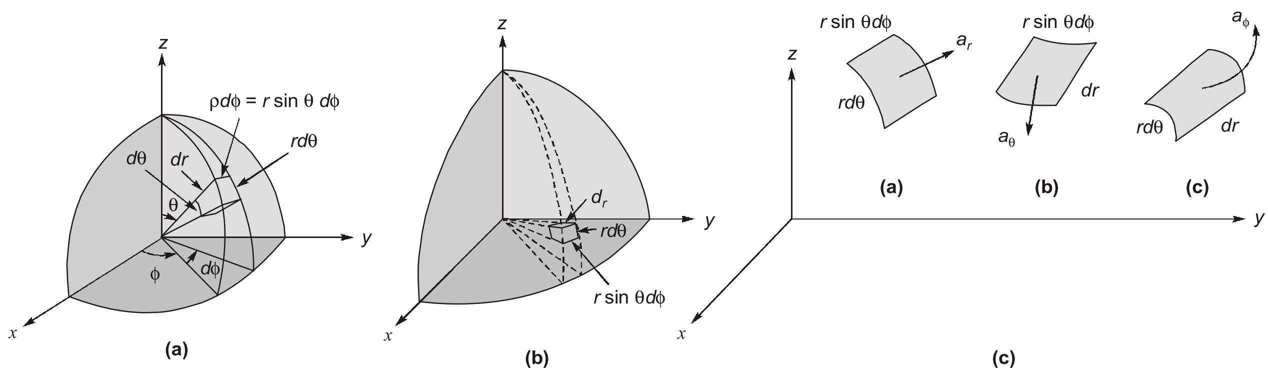


Figure. 1.9: *(a) & (b) Differential elements in spherical coordinates.*
(c) Differential normal areas in spherical coordinates

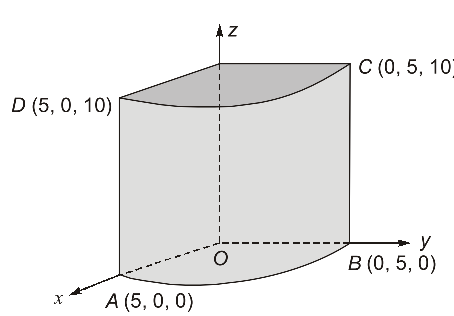
For easy reference, the differential length, surface, and volume elements for the three coordinate systems are summarized in Table 1.1.

Differential elements	Coordinate system		
	Rectangular (Cartesian)	Cylindrical	Spherical
Length $d\vec{l}$	$dx\,\vec{a}_x$ $+dx\,\vec{a}_y$ $+dz\,\vec{a}_z$	$d\rho\,\vec{a}_\rho$ $+\rho d\phi\,\vec{a}_\phi$ $+dz\,\vec{a}_z$	$dr\,\vec{a}_r$ $+r\,d\phi\,\vec{a}_\theta$ $+r\sin\theta\,d\phi\,\vec{a}_\phi$
Surface $\vec{ds}$	$dy\,dz\,\vec{a}_x$ $+dx\,dz\,\vec{a}_y$ $+dx\,dy\,\vec{a}_z$	$\rho d\phi\,dz\,\vec{a}_\rho$ $+d\rho\,dz\,\vec{a}_\phi$ $\rho d\rho\,d\phi\,\vec{a}_z$	$r^2\sin\theta\,d\theta\,\vec{a}_r$ $+r\,dr\sin\theta\,d\phi\,\vec{a}_\theta$ $+r\,dr\,d\theta\,\vec{a}_\phi$
Volume dv	$dx\,dy\,dz$	$\rho d\rho\,d\phi\,dz$	$r^2 dr\sin\theta\,d\theta\,d\phi$

Table 1.1 : *Differential elements of length, surface, and volume in the rectangular, cylindrical, and spherical coordinate systems*

***Example*-1.3** Consider the object shown in figure below. Calculate:

1. The distance *BC*.
2. The distance *CD*.
3. The surface area *ABCD*.
4. The surface area *ABO*.
5. The surface area *AOFD*.
6. The volume *ABDCFO*.



Solution:

Although points A, B, C and D are given in Cartesian coordinates, it is obvious that the object has cylindrical symmetry. Hence, we solve the problem in cylindrical coordinates. The points are transformed from Cartesian to cylindrical coordinates as follows:

$$A(5, 0, 0) \to A\,(5, 0°, 0)$$

$$B\,(0, 5, 0) \to A\left(5, \frac{\pi}{2}, 0\right)$$

$$C\,(0, 5, 10) \to C\left(5, \frac{\pi}{2}, 10\right)$$

$$D\,(5, 0, 10) \to D\,(5, 0°, 10)$$

1. Along BC, $dl = dz$; hence,

$$BC = \int dl = \int_0^{10} dz = 10$$

2. Along CD, $dl = \rho\, d\phi$ and $\rho = 5$, so

$$CD = \int_0^{\pi/2} \rho\, d\phi = 5\phi\Big|_0^{\pi/2} = 2.5\pi$$

3. For $ABCD$, $dS = \rho\, d\phi dz$, $\rho = 5$. Hence

$$\text{area } ABCD = \int dS = \int_{\phi=0}^{\pi/2}\int_{z=0}^{10} \rho\, d\phi dz = 5\int_{\phi=0}^{\pi/2} d\phi \int_{z=0}^{10} dz = 25\pi$$

4. For ABO, $dS = \rho\, d\phi d\rho$ and $z = 0$, so

$$\text{area } ABO = \int_{\phi=0}^{\pi/2}\int_{\rho=0}^{5} \rho\, d\phi\, d\rho = \int_{\phi=0}^{\pi/2} d\phi \int_0^5 \rho\, d\rho = 6.25\pi$$

5. For $AOFD$, $dS = d\rho\, dz$ and $\phi = 0°$, so

$$\text{area } AOFD = \int_{\rho=0}^{5}\int_{z=0}^{10} d\rho\, dz = 50$$

6. For volume $ABDCFO$, $dv = \rho\, d\phi\; dz\; d\rho$

$$v = \int dv = \int_{\rho=0}^{5}\int_{\phi=0}^{\pi/2}\int_{z=0}^{10} \rho\, d\phi\, dz\, d\rho = \int_0^{10} dz \int_0^{\pi/2} d\phi \int_0^5 \rho\, d\rho = 62.5\pi$$

1.3.3 Del Operator and Directional Derivative

The del operator, written ∇, is the vector differential operator. In Cartesian coordinates,

$$\nabla = \frac{\partial}{\partial x}\hat{a}_x + \frac{\partial}{\partial y}\hat{a}_y + \frac{\partial}{\partial z}\hat{a}_z \qquad ...(1.38)$$

This vector differential operator, otherwise known as the **gradient operator**, is not a vector in itself, but when it operates on a scalar function, a vector is obtained as result. The operator is useful in defining

1. The **gradient** of a scalar V, written, as ∇V.
2. The **divergence** of a vector $\vec{A}$, written as $\nabla .\ \vec{A}$.
3. The **curl** of a vector $\vec{A}$, written as $\nabla \times \vec{A}$.
4. The **Laplacian** of a scalar V, written as $\nabla^2 V$.

Vector identities: • $\nabla \cdot (\nabla \times A) = 0$ • $\nabla \times (\nabla f) = 0$ • $\nabla \times (\nabla \times A) = \nabla\,(\nabla \cdot A) - \nabla^2 A$

Each of these will be defined in detail in the following sections. The expressions for the del operator ∇ in a cylindrical and spherical coordinates are:

Cartesian coordinates, $$\nabla = \frac{\partial}{\partial x}\hat{a}_x + \frac{\partial}{\partial y}\hat{a}_y + \frac{\partial}{\partial z}\hat{a}_z$$

Cylindrical coordinates, $$\nabla = \frac{\partial}{\partial \rho}\hat{a}_\rho + \frac{1}{\rho}\frac{\partial}{\partial \phi}\hat{a}_\phi + \frac{\partial}{\partial z}\hat{a}_z \quad ...(1.39)$$

Spherical coordinates, $$\nabla = \frac{\partial}{\partial r}\hat{a}_r + \frac{1}{r}\frac{\partial}{\partial \theta}\hat{a}_\theta + \frac{1}{r\sin\theta}\frac{\partial}{\partial \phi}\hat{a}_\phi \quad ...(1.40)$$

1.3.4 Gradient of a Scalar

The gradient of a scalar field V is a vector that represents both the magnitude and the direction of the maximum space rate of change of V.

It depends upon the position where the gradient is to be evaluated and it may have different magnitudes and directions locations in space.

The gradient of V can be expressed in Cartesian, cylindrical, and spherical coordinates.

Cartesian coordinates, $$\nabla V = \frac{\partial V}{\partial x}\hat{a}_x + \frac{\partial V}{\partial y}\hat{a}_y + \frac{\partial V}{\partial z}\hat{a}_z \quad ...(1.41)$$

Cylindrical coordinates, $$\nabla V = \frac{\partial V}{\partial \rho}\hat{a}_\rho + \frac{1}{\rho}\frac{\partial V}{\partial \phi}\hat{a}_\phi + \frac{\partial V}{\partial z}\hat{a}_z \quad ...(1.42)$$

Spherical coordinates, $$\nabla V = \frac{\partial V}{\partial r}\hat{a}_r + \frac{1}{r}\frac{\partial V}{\partial \theta}\hat{a}_\theta + \frac{1}{r\sin\theta}\frac{\partial V}{\partial \phi}\hat{a}_\phi \quad ...(1.43)$$

NOTE

- $\nabla (U + V) = \nabla U + \nabla V$
- $\nabla (UV) = V\nabla U + U\nabla V$
- If $\vec{A} = \nabla V$, V is said to be the scalar potential of A.

1.3.5 Divergence of a Vector and Divergence Theorem

The divergence of $\vec{A}$ at a given point P is the outward flux per unit volume as the volume shrinks about P.

Hence, $$\text{div } \vec{A} = \nabla \cdot \vec{A} = \lim_{\Delta v \to 0} \frac{\oint_S \vec{A}.\vec{dS}}{\Delta v} \quad ...(1.45)$$

Where Δv is the volume enclosed by the closed surface S in which P is located. Physically, we may regard the divergence of the vector field A at a given point as a measure of how much the field diverges or emanates from that point. Figure 1.12 (a) shows that the divergence of a vector field at point P is positive because the vector diverges (or spreads out) at P. In figure 1.12 (b) a vector field has negative divergence (or convergence) at P, and in figure 1.12 (c) a vector field has zero divergence at P.

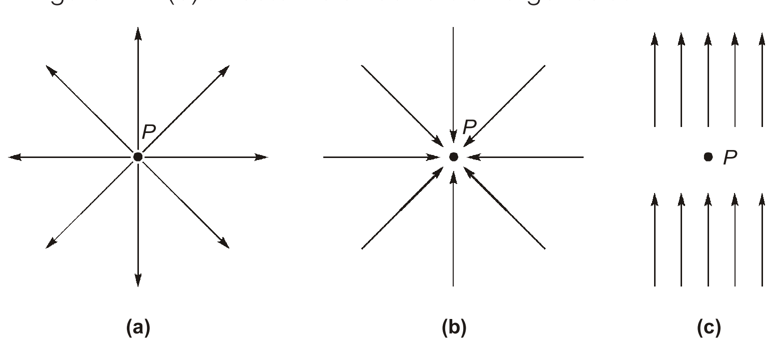


Figure 1.12: *Illustration of the divergence of a vector field at P; (a) positive divergence, (b) negative divergence, (c) zero divergence.*

The divergence of $\vec{A}$ at point P is given by:

Cartesian coordinates $$\nabla \cdot \vec{A} = \frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial z} \quad ...(1.46)$$

Cylindrical coordinates $$\nabla \cdot \vec{A} = \frac{1}{\rho}\frac{\partial}{\partial \rho}(\rho A_\rho) + \frac{1}{\rho}\frac{\partial A_\phi}{\partial \phi} + \frac{\partial A_z}{\partial z} \quad ...(1.47)$$

Spherical coordinates $$\nabla \cdot \vec{A} = \frac{1}{r^2}\frac{\partial}{\partial r}(r^2 A_r) + \frac{1}{r\sin\theta}\frac{\partial}{\partial \theta}(A_\theta \sin\theta) + \frac{1}{r\sin\theta}\frac{\partial A_\phi}{\partial \phi} \quad ...(1.48)$$

From the definition of the divergence of A, we can write that

$$\oint_S \vec{A}\cdot d\vec{S} = \int_v \nabla.\vec{A}\,dv \quad ...(1.49)$$

This is called the divergence theorem.

Remember: The divergence theorem states the total outward flux of a vector field $\vec{A}$ through the closed surface S is the same as the volume integral of the divergence of A.

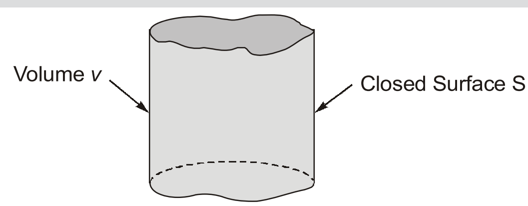
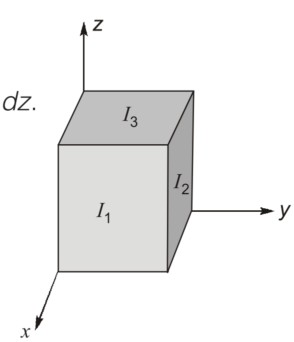


Figure 1.13: *Volume v enclosed by surface S.*

Example - 1.4 For a vector field $\vec{A} = xyz^2\hat{a}_x = xy^2z\hat{a}_y + x^2yz\hat{a}_y$. Evaluate the surface integral for a surface of unit cube defined by $0 \le x \le 1$, $0 \le y \le 1$, $0 \le z \le 1$.



Solution:

Given unit cube, therefore, the limits of integration are 0 to 1 for dx, dy and dz.

$$\oint \vec{A}\cdot d\vec{S} = \int_0^1\int_0^1 xyz^2 dydz + \int_0^1\int_0^1 xy^2 zdxdz + \int_0^1\int_0^1 x^2yzdxdy$$

$$= I_1 + I_2 + I_3$$

$$I_1 = -\int_0^1\int_0^1 0.dy\,dz + \int_0^1\int_0^1 yz^2 dydz = \frac{1}{2}.\frac{1}{3} = \frac{1}{6}$$

$$I_2 = \int_0^1\int_0^1 xz\,dx\,dz = \frac{1}{2}\frac{1}{2} = \frac{1}{4} \quad \text{and} \quad I_3 = \int_0^1\int_0^1 x^2y\,dx\,dz = \frac{1}{3}.\frac{1}{2} = \frac{1}{6}$$

$$\oint \vec{A}\cdot d\vec{S} = \frac{1}{6} + \frac{1}{4} + \frac{1}{6} = \frac{7}{12}$$

Example - 1.5 Verify the above example result by using the divergence theorem.

Solution:

According to the divergence theorem.

$$\oint \vec{A}\cdot d\vec{S} = \int_v (div\ \vec{R})dv$$

$$div\ \vec{A} = \frac{dA}{dx} + \frac{dA}{dy} + \frac{dA}{dz} = yz^2 + 2yzx + x^2y = \iiint_0^1 (yz^2 + 2yzx + x^2y)dx\,dy\,dz$$

$$= \int_0^1\int_0^1\left(\frac{y}{3}+xy+x^2y\right)dx\,dy = \int_0^1\left(\frac{y^2}{6}+\frac{xy^2}{2}+\frac{x^2y^2}{2}\right)_0^1 dx$$

$$= \int_0^1\left(\frac{1}{6}+\frac{x}{2}+\frac{x^2}{2}\right)dx = \frac{1}{6}+\frac{1}{4}+\frac{1}{6} = \frac{7}{12}$$

Hence, $\oint \vec{A}\cdot d\vec{S} = \int_v (div\,\vec{A}) = \frac{7}{12}$

1.3.6 Curl of a Vector and Stokes's Theorem

The curl is a vector operation that can be used to state whether there is a rotation associated with a vector field.

The curl of $\vec{A}$ is an axial (or rotational) vector whose magnitude is the maximum circulation of A per unit area as the area tends to zero and whose direction is the normal direction of the area when the area is oriented so as to make the circulation maximum.

That is,
$$\text{curl } \vec{A} = \nabla\times\vec{A} = \left(\lim_{\Delta S\to 0}\frac{\oint_L \vec{A}.\vec{dl}}{\Delta S}\right)_{max}\hat{a}_n \qquad ...(1.50)$$

Where the area ΔS is bounded by the curve L and $\hat{a}_n$ is the unit vector normal to the surface ΔS and is determined using the right-hand rule.

The culr of A at point P is given by:

Cartesian coordinates
$$\nabla\times\vec{A} = \begin{vmatrix}\hat{a}_x & \hat{a}_y & \hat{a}_z\\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z}\\ A_x & A_y & A_z\end{vmatrix} \qquad ...(1.51)$$

Cylindrical coordinates
$$\nabla\times\vec{A} = \frac{1}{\rho}\begin{vmatrix}\hat{a}_\rho & \rho\hat{a}_\phi & \hat{a}_z\\ \frac{\partial}{\partial \rho} & \frac{\partial}{\partial \phi} & \frac{\partial}{\partial z}\\ A_\rho & \rho A_\phi & A_z\end{vmatrix} \qquad ...(1.52)$$

Spherical coordinates
$$\nabla\times\vec{A} = \frac{1}{r^2\sin\theta}\begin{vmatrix}\hat{a}_r & r\hat{a}_\theta & r\sin\theta\hat{a}_\phi\\ \frac{\partial}{\partial r} & \frac{\partial}{\partial \theta} & \frac{\partial}{\partial \phi}\\ A_\rho & rA_\theta & r\sin\theta A_\phi\end{vmatrix} \qquad ...(1.53)$$

The physical significance of the curl of a vector field is evident; the curl provides the maximum value of the circulation of the field per unit area (or circulation density) and indicates the direction along which this maximum value occurs. The curl of a vector field $\vec{A}$ at a point P may be regarded as a measure of the circulation or how much the field curls around P. For example, figure 1.14 (a) shows that the curl of a vector field around P is directed out of the page. Figure 1.14 (b) shows a vector field with zero curl.

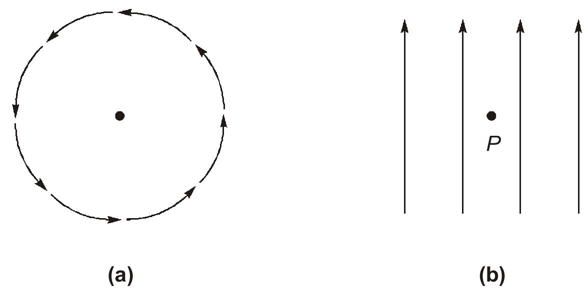


***Figure 1.14:** Illustration of a curl: (a) curl at P points out of the page; (b) curl at P is zero.*

From the definition of the curl of a vector we can obtain Stokes' theorem that relates a closed line integral to a surface integral. We can write

$$\oint_L \vec{A} \cdot \vec{dl} = \int_S (\nabla \times \vec{A}).d\vec{S} \qquad ...(1.54)$$

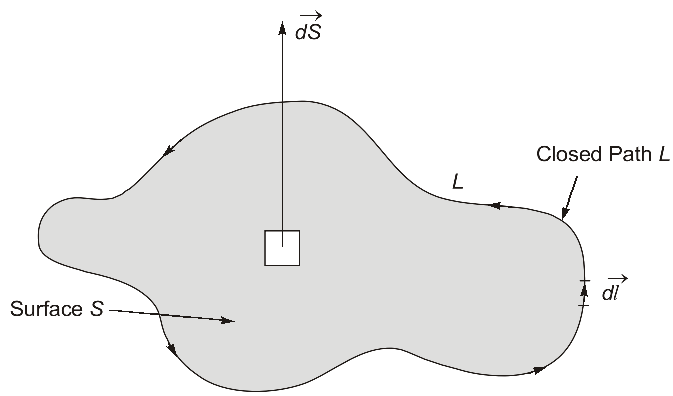


***Figure 1.15:** Determining the sense of $\vec{dl}$ and $\vec{dS}$ involved in Stokes's theorem.*

Remember: Stokes's theorem states that the circulation of a vector field $\vec{A}$ around a (closed) path L is equal to the surface integral of the curl of $\vec{A}$ over the open surface S enclosed by loop L provided that $\vec{A}$ and $\nabla \times \vec{A}$ are continuous on S.

Example - 1.6 Given vector $\vec{A} = x^2 y\hat{a}_x + 2xy^2\hat{a}_y$, find circulation of $\vec{A}$ along a closed path *OABC* as shown in figure below.